INVENTORY MODELS WITH SUBSTITUTION

By

SUNIL KUMAR NIGAM

INDUSTRIAL AND MANAGEMENT ENGINEERING PROGRAMME

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SUNIL KUMAR NIGAM

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CERTIFICATE

This is to certify that the present work on "Inventory Models with Substitution", by Sunil Kumar Nigam, has been carried out under my supervision and has not been submitted elsewhere for the award of a degree.

Kripa Shanker

(Kripa Shanker)
Assistant Professor
Industrial and Management Engineering
Indian Institute of Technology
Kanpur-16

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SUNIL KUMAR NIGAM

CONTENTS

ABSTRACT

The present work on substitution deals with various models where the interaction of demand due to shortages and/or introduction of new items is taken into account in two-item/multi-item inventory system.

Deterministic and stochastic models have been developed for various situations. In deterministic case, models for independent procurement, joint procurement of two-item inventory system have been developed for complete substitution and partial substitution of an item by the other in stock out situation. The coordinated replenishment policy is also considered for two-item case with substitution. Models for continuous substitution of demand of old product by the new product in two-item inventory system have developed for the situations allowing shortages and no shortages. The analysis of dynamic demand case, to take account the seasonality in demands of item and fluctuation in cost parameters, has been carried out to decide the procurement policy in two-item inventory system using the concepts of dynamic programming. The model for the complete substitution of the demand in n-item inventory system has been developed. The joint replenishment model for stochastic demand case has been developed for periodic review order-up-to-level (R, T) policy. The simulation has been carried out for (Q, r, T) policy.

The solution methodology has been suggested in all cases. Illustrative examples are given to study the effect of substitution on ordering policy for various models. It was observed that the ordering policy are substantially different from that when the items are treated independently. In certain cases, there is substantial reduction in the operating cost when the substitution is permitted.

CHAPTER I

INTRODUCTION

1.1 Inventory Control and Substitution - An Over View:

Organizations maintain inventories of raw materials, semi-finished and finished goods in order to minimize the effects of uncertainty of demand and production lead time. Raw materials inventories are used as input to the production system, and finished goods inventories are maintained to satisfy customer demands. Like other investments, inventories blockup invested capitals. In certain industries the amount lockedup in inventories can be as high as 25 percent of the total investment, while in others it may go upto 80 percent of the total.

There is always a growing concern in any organization to improve the customers satisfaction. At the same time, the organization would like to have the best possible return on investment through proper utilization of money in various aspects of inventory management. In several situations items can be used to satisfy the demands of some other items with similar characteristics such phenomenon of substitution is prevalent in many other situations. In literature the term substitution is used for the replacement of technology, style, method, equipment, material customer goods, industrial products, etc.

The substitution or replacement of the technology is considered for long range planning. This is dependent more on the research work and inventions. Different existing and under process technologies can be compared to take alternate decisions for the selection of the appropriate technology for future. In manufacturing process, the old and obselete machine and equipments are substituted by new ones. In short range planning we can consider the different similar type of technologies, equipments, machines or products for the substitution of one by other. The decision for their selection is taken on the basis of utility, availability, reliability and market demand of the products. In immediate planning horizon we consider industrial products having similar functional requirement, quality etc. for the substitution.

Many industrial products have either one way or two way substitution.

<u>One Way Substitution:</u>

One way substitution implies that a class of item can substitute for the other class but not vice versa because of technological limitations. For example, a high strength steel beam may be substituted for the a class of lower strength beam, but not other way.

<u>Two Way Substitution:</u>

In many industrial situation there are no as sharp

technological constraints as with the case of the one-way substitution. The products or materials having similar or very close functional characteristics can be considered for the substitution of the demand of each other.

In addition to technological functional characteristics, the customer behavior also plays an important role in resupplying the demands. The customer sometimes has preference to purchase specific type of products. This preference depends on various factors like need, price, quality, availability of the products etc. This gives rise to different ranks in terms of utility value for different similar types of products. In the event of shortages of high utility value product the customer needs can be fulfilled by supplying the next available utile value product. Thus the demands of one product are substituted by the other products and thus there exists an interaction among the demands of different products.

Apart from the technological constraints and consumer behavior, the economic factor play an important role in deciding the substitution of one with other.

Even though the substitution aims at minimizing the loss of good will by making a similar item available to the customer in place of one originally demanded, there are costs associated with the process of substitution. The substitution costs may appear to the system in the form of decrease in the

unit profit of substituting item which when supplied in response of the right demand would have fetched a higher profit, handling charges re-work charges etc. Furthermore relative difference with various costs such as purchase price holding, procurement cost etc. may decide the operating inventory level of various item and interaction of their demands.

The objective here like in any other inventory system is to establish the operating policy such that total operating cost is a minimum. The operating policy now includes the decision on level of substitution in addition to usual decision of how much to procure and when to procure. And the total operating cost include substitution in addition to the procurement and holding costs.

1.2 Literature Review:

The vast literature on inventory addresses the single item inventory system. In last two decade there has been appreciable attempt to study the multi-item inventory systems. But most of the mathematical model have been developed without considering the interactions among the items. However, in many real life situations interaction among items do exist in terms of substitution.

Even though many researchers and authers ofthe text books on inventory system such as Hadley and Whitin [1] have stressed upon the importance of substitutability of items for inclusion

in development of inventory models, no appreciable attempts has been paid in terms of policy formulation. Ruterberg [2] has investigated one way and two way substitutions for considering commonality of demand and has formulated as a transportation problem. Pentico [3] has discussed the substitution of the group of small size by another group of larger size in carrying extra cost as a substitution cost, in probabilistic demand case. Townsend [4] has considered the case of n-items inventory system where few of items among all may be selected to satisfy the demand of all items. He has assumed the cost of placing an order is negligible at the same time he has not considered the relative importance of cost structure. Krishanan [5] has considered the case where the demand in shortage can be satisfied by other items. He has used dynamic programming approach for deciding optimal quantities of items. Mcgillivray and Silver [6] have investigated the effects of substitutable demand on stocking control rules and associated inventory/shortage costs in probabilistic demand case. They have developed the model for limiting case in which all substitutable items have the same unit variable cost and shortage penality.

1.3 Organization of Thesis:

The present study deals with multi-item inventory system under substitutable demand consideration. The substitution of

demand of an item in stock out condition by the other, for two items case has been dealt in Chapter II. In this chapter we assume for complete substitution of demand. Here we have taken the general case where all costs parameters differ. The problem of partial substitution with backlogging for two items has been dealt in Chapter III. Chapter IV deals the problem of joint replenishment with end substitution. In this chapter the cases related to the complete as well as partial substitution have been dealt. Chapter V deals with the joint replenishment of two items with continuous substitution of an item by a newly introduced item. In this chapter we deal the case of no shortages as well as the case when shortages are partially backlogged. Chapter VI deals the coordinated (mixed) replenishment policy with complete substitution of demand of an item by the other. Chapter VII deals the problem of substitution of two items in case of dynamic demand. The concept of dynamic programming is used to arrive at the optimal solution. Chapter VIII deals the problem of several items where the demand of an item is completely substituted by other item. Various cases have been dealt to decide the procurement policy. The case of joint replenishment for stochastic demand - Periodic Review order-up-to (R.T.) policy has been discussed in Chapter IX. In Chapter X (Q,r,T) model with coordinated replenishment has been dealt. The problem has been solved using simulation

technique. Chapter XI offers some concluding remarks and put forward some suggestions for further work.

CHAPTER II

PLANNED SUBSTITUTION OF TWO ITEMS

2.1 Problem Statement:

Consider an inventory system of a retailer/whole sale stocking. The system is operating with large number of items. Among them, few items have similar characteristics in terms of their utilities, purchase costs etc. and can be put in one group. We now concentrate on a group of two items which can substitute each other. It is assumed here that in the situation of stock out of one item its demand can be completely satisfied by the other. And there will be atleast one item in the stock at any time to avoid shortages.

A cost of substitution is incurred when a customer demands for an item is substituted by the other in the stock out of the first.

For the situation described above we shall find out the optimal stocking policies for the two items such that the total cost per unit time (year) of the system is minimized. The total cost would include usual holding and procurement (fixed and variable) cost and cost of substitution.

2.2 Assumptions:

The following assumptions are made in order to further characterize the above described inventory system of two substitutable items.

1) The demand rate for each item is deterministic and static.
2) The replenishment is instantaneous.
3) The unsatisfied demand of an item is completely fulfilled by the other. Thus shortage for any item is not allowed.
4) The planning horizon is infinite.
5) Unit variable cost of any item does not depend upon the replenishment quantities. In other words there is no quantity discount.
6) The substitution cost is directly proportional to the number of units to be substituted.

2.3 Nomenclature:

Parameters:

D_1 Annual demand rate of item 1.

D_2 Annual demand rate of item 2.

h_1 Holding cost of item 1 in Rs/Unit/Unit time.

h_2 Holding cost of item 2 in Rs/Unit/Unit time.

C_1 Unit cost of item 1.

C_2 Unit cost of item 2.

A_1 Fixed ordering cost of item 1.

A_2 Fixed ordering cost of item 2.

V_1 Substitution cost of item 1 when it is substituted by item 2 in Rs./unit

V_2 Substitution cost of item 2 when it is substituted by item 1 in Rs./unit.

Decision Variables:

Q_1 The replenishment quantity of item 1 per cycle in units.

Q_2 The replenishment quantity of item 2 per cycle in units.

S_1 The inventory level of item 1 at the time of the procurement of the item 2.

S_2 The inventory level of item 2 at the time of replenishment of item 1.

Objective Function:

TC Total annual cost of both items

Cycle Time T The time interval between two consecutive replenishments of the same item. This is same for both the items.

Other intermediate notations will be introduced during formulation.

2.4 Formulation:

The inventory levels of items 1 and 2 are shown in Fig. 2.1. Referring to the figure, we see that at the beginning of the cycle, Q_1 units of item 1 are replenished and S_2 units of item 2 are on hand. The amount S_2 depletes in time T_1 at the rate D_2. The next procurement of item 2 is after a time T_2. Thus item 1 substitutes the demand for item 2 for the duration T_2. Therefore, as shown in the figure, units of item 1 deplete at the rate of $(D_1 + D_2)$ during the time T_2. At the end of T_2, S_1 units of item 1 are on hand, and Q_2 amount of item 2 is replenished. Again in time T_3, the amount S_1 depletes with rate D_1 and Q_2 with rate D_2. In time T_4, item 2 depletes with the demand rate of D_1+D_2 and thus item 2 substitutes the demand for item 1 for the duration T_4. At the end of T_4, S_2 units of item 2 are left and item 1 is received in Q_1 units. And thus cycle repeats.

From the figure, following relations can be easily derived.

$$T_1 = S_2/D_2 \quad \text{(Time in a cycle in which item is satisfies its own demand)} \qquad (2.1)$$

$$T_2 = \frac{Q_1 - D_1T_1 - S_1}{D_1+D_2} \quad \text{(Time in a cycle in which item 1 satisfies the demand of items 1 and 2)} \qquad (2.2)$$

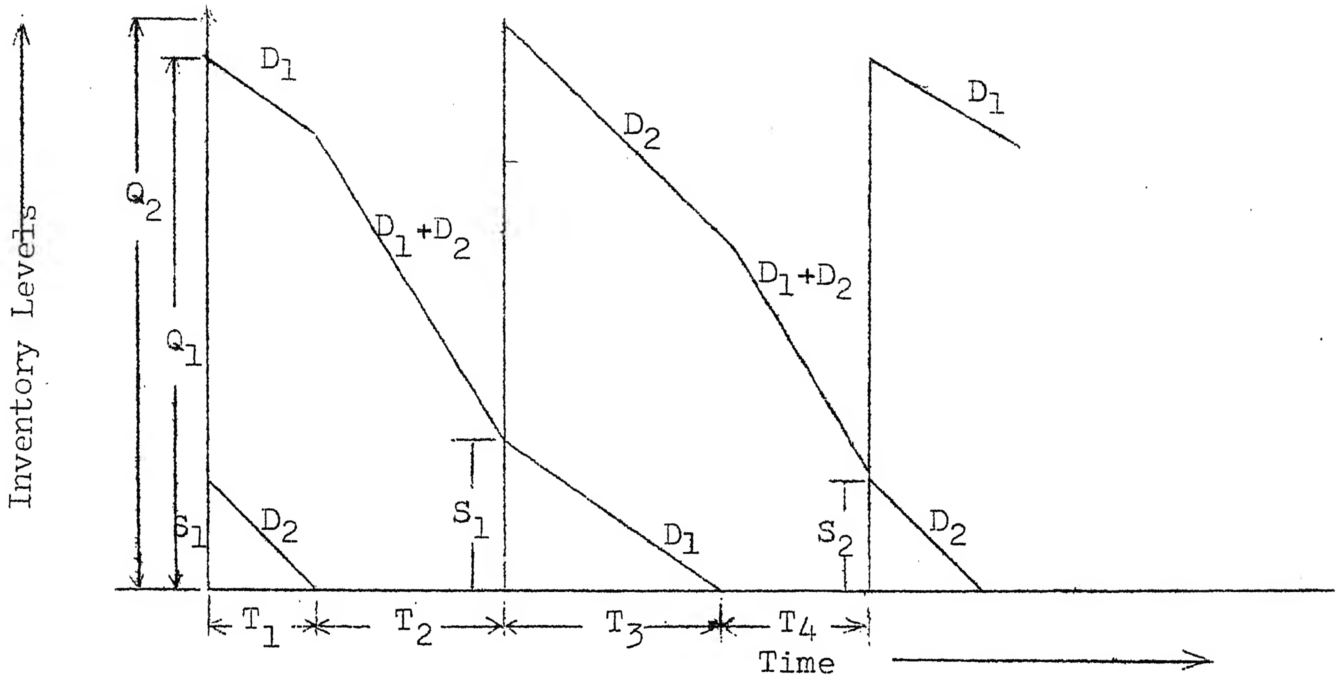

Fig. 2.1: Substitution of two items with no backlogging (General Case)

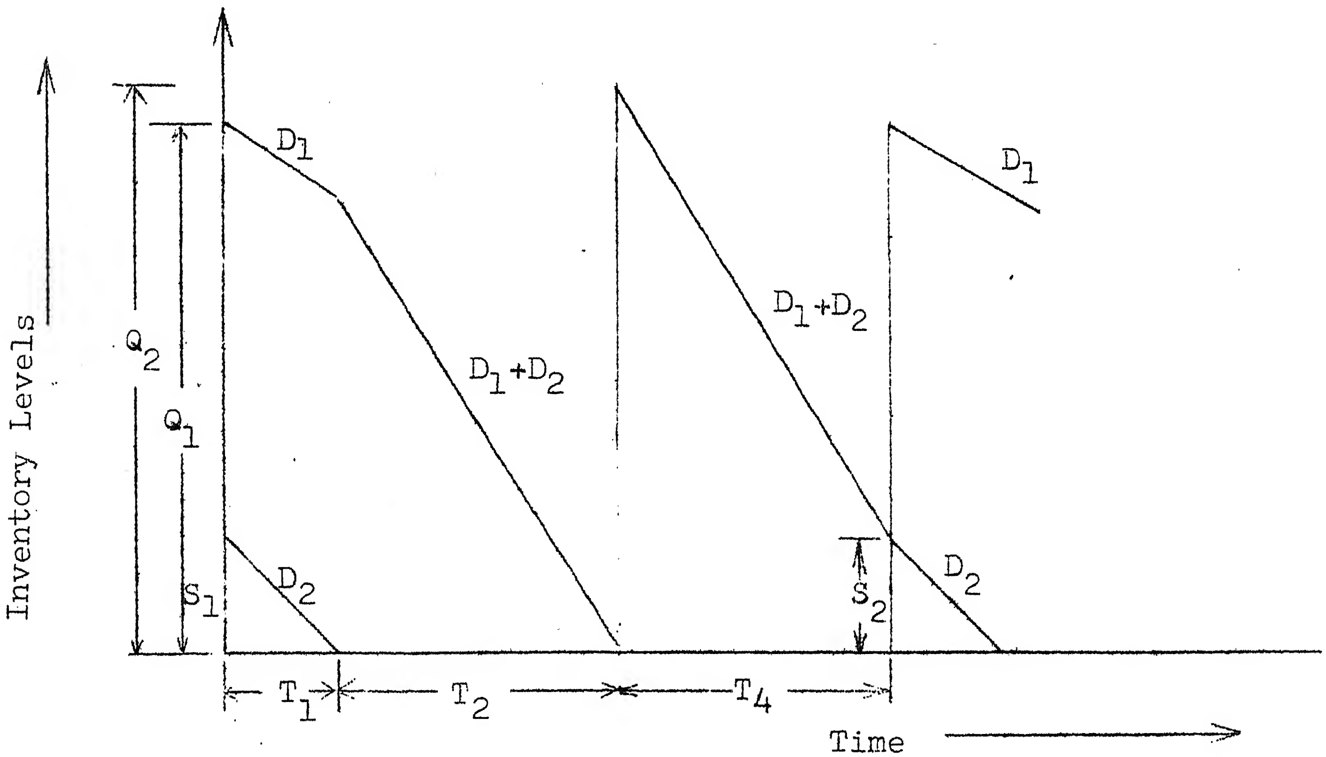

Fig. 2.2: Substitution of two items with no backlogging (Case : I).

$$T_3 = S_1/D_1 \tag{2.3}$$

$$T_4 = \frac{Q_2 - D_2 T_3 - S_2}{(D_1 + D_2)} \tag{2.4}$$

Thus, cycle time $T = T_1 + T_2 + T_3 + T_4$ (2.5)

Substituting T_1, T_2, T_3 and T_4 from Eqs. (2.1) - (2.4) in Eqn. (2.5), we have,

$$T = \frac{S_2}{D_2} + \frac{Q_1 - D_1.(S_2/D_2) - S_1}{(D_1+D_2)} + \frac{S_1}{D_1} + \frac{Q_2 - D_2.(S_1/D_1) - S_2}{(D_1+D_2)}$$

$$= \frac{(Q_1+Q_2)}{(D_1+D_2)} + S_2\left[\frac{1}{D_2} - \frac{D_1}{D_2(D_1+D_2)} - \frac{1}{(D_1+D_2)}\right]$$

$$+ S_1\left[\frac{1}{D_1} - \frac{1}{(D_1+D_2)} - \frac{D_2}{D_1(D_1+D_2)}\right]$$

$$= \frac{(Q_1+Q_2)}{(D_1+D_2)} \tag{2.6}$$

Various relevant costs per cycle are given, as follows.

(A) Procurement Cost:

For item 1 $= A_1 + C_1 Q_1$ (2.7)

For item 2 $= A_2 + C_2 Q_2$ (2.8)

(B) Holding Cost:

For item 1 $= h_1$ (area under inventory of item 1)

$$= h_1 \frac{T_1}{2}(Q_1 + Q_1 - D_1 T_1) + \frac{h_1 T_2}{2}(Q_1 - D_1 T_1 + S_1)$$

$$+ \frac{h_1 T_3 S_1}{2}$$

$$= \frac{Q_1 S_2 h_1}{(D_1+D_2)} - \frac{S_2^2 h_1 D_1}{2D_2(D_1+D_2)} + \frac{Q_1^2 h_1}{2(D_1+D_2)}$$

$$+ \frac{S_1^2 h_1 D_2}{2D_1(D_1+D_2)} \qquad (2.9)$$

For item 2 $= h_2 \times$ (area under inventory of item 2)

$$= \frac{h_2 T_3}{2}(2Q_2 - D_2T_3) + \frac{h_2 T_4}{2}(Q_2 - D_2T_3 + S_2)$$

$$+ \frac{h_2 S_2 T_1}{2}$$

$$= \frac{Q_2 S_1 h_2}{(D_1+D_2)} - \frac{S_1^2 h_2 D_2}{2D_1(D_1+D_2)} + \frac{Q_2^2 h_2}{2(D_1+D_2)}$$

$$+ \frac{S_2^2 h_2 D_1}{2D_2(D_1+D_2)} \qquad (2.10)$$

(C) Substitution Cost:

For item 2 $= V_2 D_2 T_2$

$$= \frac{V_2 D_2}{(D_1+D_2)}\left(Q_1 - \frac{D_1}{D_2} S_2 - S_1\right) \qquad (2.11)$$

For item 1 $= V_1 D_1 T_4$

$$= \frac{V_1 D_1}{(D_1+D_2)}\left(Q_2 - \frac{D_2 S_1}{D_1} - S_2\right) \qquad (2.12)$$

The total cost per cycle will be the sum of procurement holding and substitution costs. Therefore, the total annual cost is given by,

$$TC = \frac{1}{T}\Big[\frac{Q_1S_2h_1}{(D_1+D_2)} - \frac{S_2{}^2h_1D_1}{2D_2(D_1+D_2)} + \frac{Q_1{}^2\,h_1}{2(D_1+D_2)} + \frac{S_1{}^2\,h_1D_2}{2D_1(D_1+D_2)}$$

$$+ \frac{Q_2S_1h_2}{(D_1+D_2)} - \frac{S_1{}^2\,h_2D_2}{2D_1(D_1+D_2)} + \frac{Q_2{}^2\,h_2}{2(D_1+D_2)} + \frac{S_2{}^2h_2D_1}{2D_2(D_1+D_2)}$$

$$+ \frac{V_2D_2}{(D_1+D_2)}\,(Q_1 - \frac{D_1S_2}{D_2} - S_1) + \frac{V_1D_1}{(D_1+D_2)}\,(Q_2 - \frac{D_2}{D_1}\,S_1 - S_2$$

$$= \frac{1}{(Q_1+Q_2)}\,[Q_1S_2h_1 + \frac{S_2{}^2D_1}{2D_2}\,(h_2-h_1) + \frac{Q_1{}^2h_1}{2} + \frac{S_1{}^2D_2}{2D_1}$$

$$x\,(h_1-h_2) + Q_2S_1h_2 + \frac{Q_2{}^2h_2}{2} + \{C_1(D_1+D_2)+V_2D_2\}\,Q_1$$

$$+ \{C_2(D_1+D_2) + V_1D_1\}\,Q_2-(V_1+V_2)\,D_1S_2$$

$$- (V_1+V_2)D_2S_1 + (A_1+A_2)(D_1+D_2)] \qquad (2.13)$$

Differentiating Eqn. (2.13) with respect to decision variables, Q_1, Q_2, S_1 and S_2 and setting the derivatives equal to zero, we get,

$$\frac{\partial\,TC}{\partial\,Q_1} = \frac{1}{(Q_1+Q_2)}\,[S_2h_1 + Q_1h_1 + C_1(D_1+D_2) + V_2D_2]$$

$$- \frac{TC}{(Q_1+Q_2)^2} = 0 \qquad (2.14)$$

$$\frac{\partial\,TC}{\partial\,Q_2} = \frac{1}{(Q_1+Q_2)}\,[S_1h_2 + Q_2h_2 + C_2(D_1+D_2) + V_1D_1]$$

$$- \frac{TC}{(Q_1+Q_2)^2} = 0 \qquad (2.15)$$

$$\frac{\partial TC}{\partial S_1} = \frac{1}{(Q_1+Q_2)} [Q_2h_2 + S_1 \frac{D_2}{D_1} (h_1-h_2) - (V_1+V_2) D_2] = 0 \tag{2.16}$$

$$\frac{\partial TC}{\partial S_2} = \frac{1}{(Q_1+Q_2)} [Q_1h_1 + S_2 \frac{D_1}{D_2} (h_2-h_1) - (V_1+V_2) D_1] = 0 \tag{2.17}$$

The solution obtained from above equations can be global minimum, only if second derivatives with respect to all these variable are positive. Since it is difficult to derive some conclusions from second derivatives with respect to Q_1 and Q_2, we check the conditions of second derivatives with respect to S_1 and S_2. The second derivatives with respect to S_1 and S_2 are

$$\frac{\partial^2 TC}{\partial S_1^2} = \frac{D_2}{D_1} (h_1 - h_2) \tag{2.18}$$

$$\frac{\partial^2 TC}{\partial S_2^{\,2}} = \frac{D_1}{D_2} (h_2 - h_1) \tag{2.19}$$

There are two possible conditions.

1. When $h_1 > h_2$,

$$\frac{\partial^2 TC}{\partial S_1^2} > 0 \text{ , but } \frac{\partial^2 TC}{\partial S_2^2} < 0 \tag{2.20}$$

This gives an indication that for the solution to be global minimum, S_2 should either of the extreme points

(S_2 = 0 or $S_2 = \infty$). Since $S_2 = \infty$ is not a practical solution, S_2 must be zero for the optimal solution. Now this problem is exactly same, as Case II and solution procedure will be discussed in Sec. (2.4.2).

2. When $h_1 < h_2$:

$$\frac{\partial^2 TC}{\partial S_1^{\,2}} < 0 \quad \text{and} \quad \frac{\partial^2 TC}{\partial S_2^{\,2}} > 0 \tag{2.21}$$

which gives the condition that S_1 has to be 0 for global minimum. Now this problem is exactly same as Case I and the solution procedure will be discussed in Sec. (2.4.1).

Now we shall discuss some special case 3 of the general model discussed above.

2.4.1 Case I:

Here, we shall discuss the case when the inventory level of item 1 is zero at the time of the procurement of item 2 and item 2 satisfies demands of both the items for time interval T_4. This implies $S_1 = 0$ and $T_3 = 0$. And from Fig. 2.2, we have

$$T = T_1 + T_2 + T_4 = \frac{Q_1 + Q_2}{(D_1 + D_2)} \tag{2.22}$$

Now the total annual cost is given by,

$$TC = \frac{1}{(Q_1+Q_2)} [Q_1h_1S_2 + \frac{S_2{}^2D_1}{2D_2} (h_2-h_1) + \frac{Q_1{}^2h_1}{2}$$

$$\frac{Q_2{}^2 h_2}{2} + \{C_1(D_2+D_2)+V_2D_2\} Q_1 + \{C_2(D_2+D_2)$$

$$+ V_1D_1\} Q_2 - (V_1+V_2) D_1S_2 + (A_1+A_2)(D_1+D_2)] \quad (2.23)$$

Differentiating Eqs. (2.23) with respect to decision variables and setting equal to zero, we get,

$$\frac{\partial TC}{\partial Q_1} = \frac{1}{(Q_1+Q_2)} [h_1S_2 + Q_1h_1 + C_1(D_1+D_2) + V_2D_2]$$

$$- \frac{TC}{(Q_1+Q_2)^2} = 0 \quad (2.24)$$

$$\frac{\partial TC}{\partial Q_2} = \frac{1}{(Q_1+Q_2)} [Q_2h_2 + C_2 (D_1+D_2) + V_1D_1]$$

$$- \frac{TC}{(Q_1+Q_2)^2} = 0 \quad (2.25)$$

$$\frac{\partial TC}{\partial S_2} = \frac{1}{(Q_1+Q_2)} [Q_1h_1 + \frac{S_2D_1}{D_2} (h_2-h_1) -(V_2+V_1) D_1] = 0 \quad (2.26)$$

Solving Eqs. (2.24) - (2.26), we get Q_1, Q_2 in terms of S_2 as follows,

$$Q_1 = \frac{1}{\beta} [\gamma D_1 - \frac{1}{\alpha} (1-\beta) S_2] \quad (2.27)$$

$$Q_2 = [\beta - \frac{1}{\alpha} (1-\beta)] S_2 + \delta (1+\alpha) D_1 \quad (2.28)$$

and,

$$S_2^2\left[\frac{(1-\beta)^2}{2\alpha^2\beta} + \frac{1}{2} [\beta - \frac{(1-\beta)}{\alpha}]^2 - \frac{(1-\beta)}{2\alpha} \right]$$

$$+ S_2 \left[\gamma D_1 - \frac{rD_1(1-\beta)}{\alpha\beta} + \delta (1+\alpha) D_1 \{\beta - \frac{(1-\beta)}{\alpha} \}\right]$$

$$+ [\frac{1}{2} \delta^2(1+\alpha)^2 D_1 + \gamma^2 D^2 /2\beta - \nu(1+\alpha) D_1] = 0 \quad (2.29)$$

We can rewrite Eq. (2.29) as

$$ES_2^2 + FS_2 + G = 0 \quad (2.30)$$

where,

$$E = \frac{(1-\beta)^2}{2\alpha^2\beta} + \frac{1}{2} (\beta - (1-\beta)/\alpha)^2 - \frac{(1-\beta)}{2\alpha}$$

$$F = \gamma D_1 [1-\frac{(1-\beta)}{\alpha\beta}] + \delta (1+\alpha) D_1 (\beta - \frac{(1-\beta)}{\alpha})$$

$$G = \frac{1}{2} \delta^2(1+\alpha)^2 D_1^2 + \frac{\gamma^2 D_1^2}{2\beta} - \nu (1+\alpha) D_1$$

From Eq. (2.30), we get,

$$S_2 = [- F \pm (F^2-4EG)^{1/2}]/2E \quad (2.31)$$

There are two possible conditions.

(1) $F^2 < 4EG$: It implies that both the roots obtained from above equations are not real. Therefore the optimal solution should correspond to one of the extreme conditions ($S_2 = 0$, $S_2 = \infty$). Since $S_2 = \infty$ is not a practical condition, therefore we check for the other condition, $S_2 = 0$,

ii) $F^2 \geq 4EG$: Since both roots obtained from Eq. (2.31) are real, either of the following conditions can get satisfied.

(a) Both roots are negative. In this case $S_2 = 0$ will correspond to the optimal solution.

(b) Either one or both roots are positive. Since we have not proved the convexity of the objective function with respect to S_2, the roots obtained from Eq. (2.31) will correspond to either maximum or minimum value of the objective functions. The optimal solution is obtained by comparing the solution corresponding to positive roots, as obtained above and the solution corresponding to $S_2 = 0$. The solution which corresponds to minimum objective function value will be the optimal solution.

2.4.2 Case II:

Now, we consider the case when the inventory level of item 2 is zero at the time of the procurement of item 1 which satisfies demands of both the items for time interval T_2. This means,

$$S_2 = 0 \quad \text{and } T_1 = 0.$$

The inventory levels are shown in Fig. (2.3). From this we get,

$$T = T_2+T_3+T_4 = \frac{Q_1+Q_2}{(D_1+D_2)}$$

Now the total annual cost is given by,

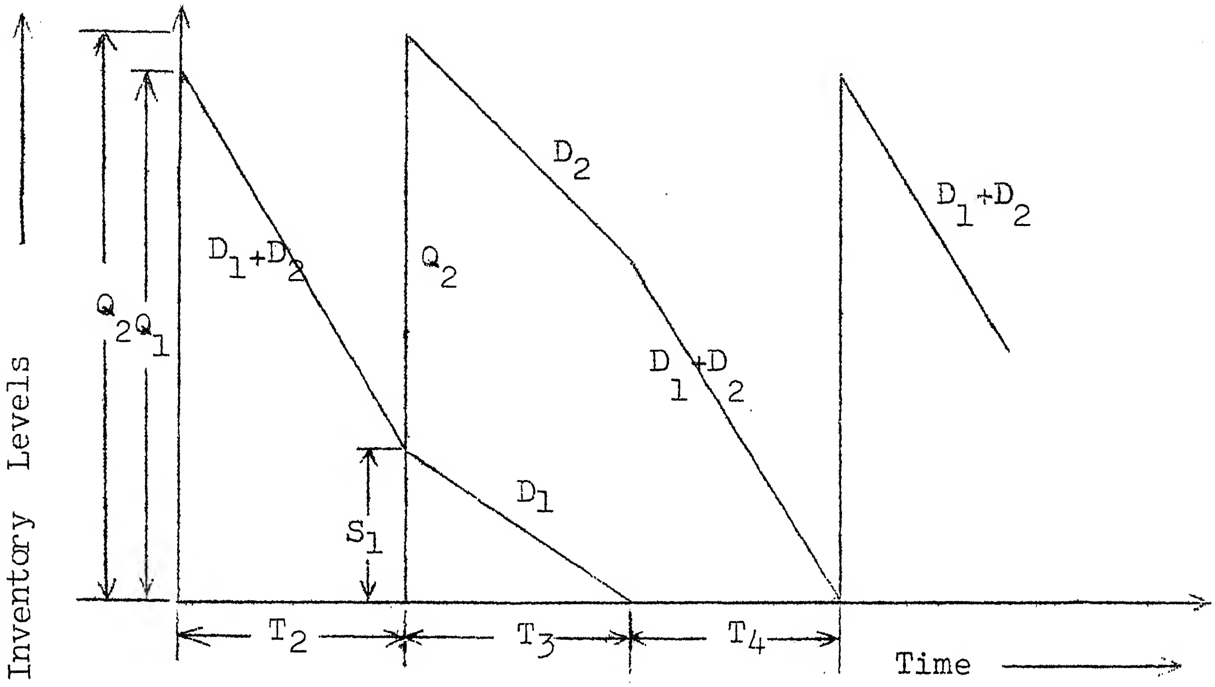

Fig. 2.3: Case II of General Model.

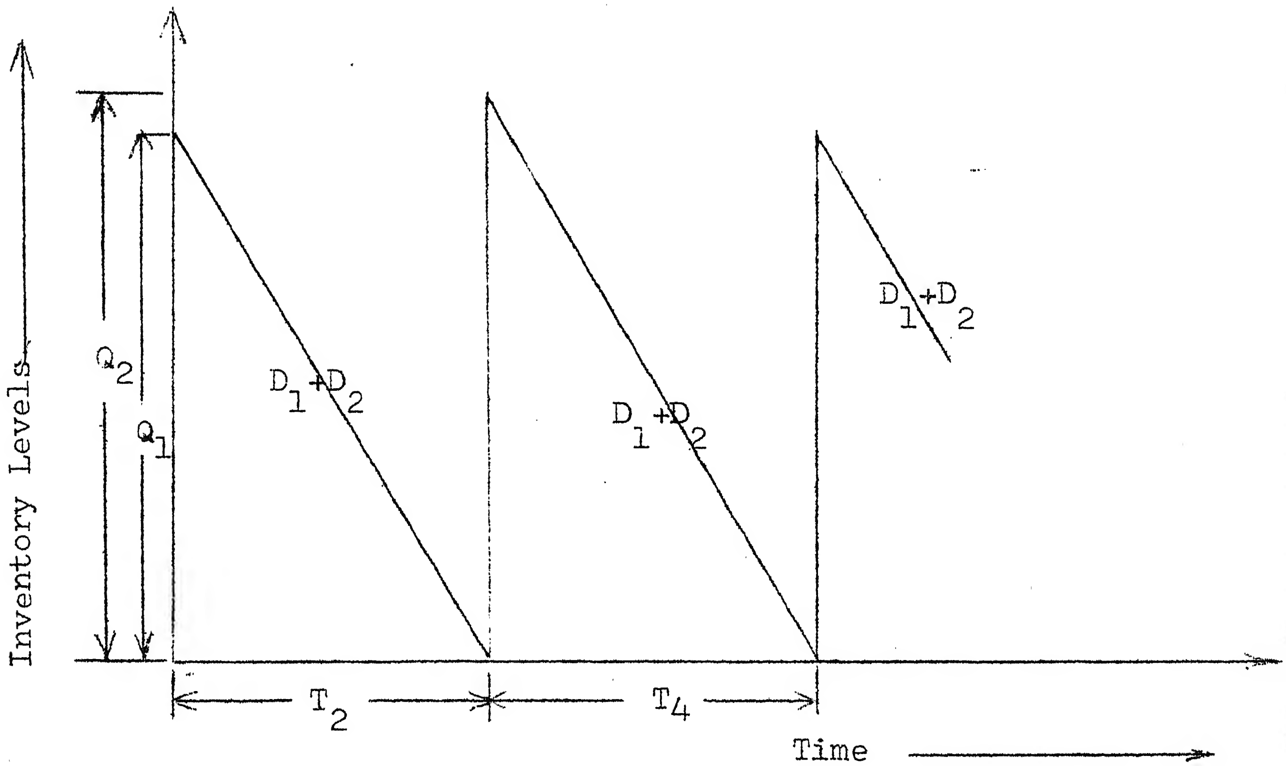

Fig. 2.4: Case III of General Model.

$$TC = \frac{1}{(Q_1+Q_2)} [\frac{Q_1{}^2 h_1}{2} + \frac{S_1{}^2 D_2}{2D_1} (h_1-h_2) + Q_2S_1h_2$$

$$+ \frac{Q_2{}^2 h_2}{2} + \{C_1(D_1+D_2) + V_2D_2\} Q_1$$

$$+ \{C_2(D_1+D_2) V_2D_1\} Q_2 - (V_1+V_2)D_2S_1$$

$$+ (A_1+A_2) (D_1+D_2)] \qquad (2.33)$$

Differentiating the Eq. (2.33), with respect to the decision variables and setting equal to zero, we get,

$$\frac{\partial TC}{\partial Q_1} = \frac{1}{(Q_1+Q_2)} [Q_1h_1 + C_1(D_1+D_2) + V_2D_2] - \frac{TC}{(Q_1+Q_2)^2} = 0 \qquad (2.34)$$

$$\frac{\partial TC}{\partial Q_2} = \frac{1}{(Q_1+Q_2)} [S_1h_2 + Q_2h_2 + C_2(D_1+D_2) + V_1D_1] - \frac{TC}{(Q_1+Q_2)^2} = 0 \qquad (2.35)$$

$$\frac{\partial TC}{\partial S_1} = \frac{1}{(Q_1+Q_2)} [Q_2 h_2 + \frac{S_1D_2}{D_1} (h_1-h_2) - (V_1+V_2) D_2] = 0 \qquad (2.36)$$

Solving Eqs. (2.34) - (2.36), we get,

$$Q_1 = \frac{1}{\beta} [S_1(1-\alpha(\beta-1) - (\delta-\gamma) (1+\alpha) D_1] \qquad (2.37)$$

$$Q_2 = \alpha\gamma D_1 - \alpha(\beta-1) S_1 \qquad (2.40)$$

and,

$$S_1^{\,2}[\{ \frac{1-\alpha(\beta-1)\}^2}{2\beta} - \alpha\ (\beta-1) + \frac{1}{2}\alpha^2\ (\beta-1)^2]$$

$$+ S_1\ [- \frac{(\delta-\gamma)(1+\alpha)D_1(1-\alpha)(\beta-1)}{\beta} + \alpha\ \beta\gamma\ D_1]$$

$$+ \frac{1}{2\beta}(\delta-\gamma)^2\ (1+\alpha)^2\ D_1^{\,2} + \frac{1}{2}\alpha^2\ \gamma^2 D_1^{\,2} - \nu(1+\alpha)\ D_1 = 0$$

$$(2.39)$$

We can rewrite Eq. (2.39) as,

$$ES_1^{\,2} + FS_1 + G = 0 \qquad (2.40)$$

where,

$$E = \{ \frac{1-\alpha(\beta-1)^2}{2\beta} \} - \alpha(\beta-1) + \frac{1}{2}\alpha^2\ (\beta-1)^2$$

$$F = - \frac{(\delta-\gamma)(1+\alpha)D_1\ (1-\alpha)(\beta-1)}{\beta} + \alpha\beta\gamma\ D_1$$

$$G = \frac{1}{2\beta}(\delta-\gamma)^2\ (1+\alpha)^2\ D_1^{\,2}\ \alpha^2\gamma^2 D_1 - \nu(1-\alpha)\ D_1$$

From Eq. (2.42), we get,

$$S_1 = [-F \pm (F^2-4EG)^{1/2}]/2E \qquad (2.41)$$

The solution procedure for obtaining the optimal solution in this case is exactly same as discussed for Case I except for the fact that S_1 should be considered instead of S_2.

2.4.3 Case III:

In this case either of the two items is procured at a time when the inventory level of the other is zero and each one satisfies the demands of both the items at any time. It means,

$S_1 = 0,\ T_3 = 0$ when item 1 is procured and

$S_2 = 0,\ T_1 = 0$ when item 1 is procured.

Fig. 2.4 shows the inventory levels of the both the items. Using this figure, we have,

$$T = T_2 + T_4$$

$$= \frac{Q_1}{D_1+D_2} + \frac{Q_2}{D_1+D_2} = \frac{Q_1+Q_2}{D_1+D_2} \qquad (2.42)$$

and the total annual cost is given by,

$$TC = \frac{1}{(Q_1+Q_2)} \left[\frac{Q_1^2 h_1}{2} + \frac{Q_2^2 h_2}{2} + \{C_1(D_1+D_2)+V_2D_2\} Q_1 + \{C_2(D_1+D_2) + V_1D_1\} Q_2 + (A_1+A_2)(D_1+D_2) \right] \qquad (2.43)$$

Differentiating Eq. (2.43) with respect to the decision variables Q_1 and Q_2 and setting them equal to zero, we get,

$$\frac{\partial TC}{\partial Q_1} = \frac{1}{(Q_1+Q_2)} [h_1Q_1 + C_1(D_1+D_2)+V_2D_2] - \frac{TC}{(Q_1+Q_2)^2} = 0 \qquad (2.44)$$

$$\frac{\partial TC}{\partial Q_2} = \frac{1}{(Q_1+Q_2)}[h_2Q_2 + C_2(D_1+D_2) + V_1D_1] - \frac{TC}{(Q_1+Q_2)^2} = 0 \tag{2.45}$$

Solving Eqs. (2.44) - (2.45), we get Q_2 in terms of Q_1

$$h_2Q_2 = h_1Q_1 + C_1(D_1+D_2) + V_2D_2 - C_2(D_1+D_2) - V_1D_1$$

or,

$$Q_2 = \beta\ Q_1 + \{\delta(1+\alpha) - \gamma\}\ D_1 \tag{2.46}$$

and,

$$\frac{1}{2}(\beta+\beta^2)\ Q_1^2 + \beta\{\delta(1+\alpha) - \gamma\}\ D_1Q_1 + \{\delta(1+\alpha) - \gamma\}^2\ D_1^2$$
$$- \nu(1+\alpha)\ D_1 = 0 \tag{2.47}$$

Finally, we get,

$$Q_1 = [-\beta\ \{\delta(1+\alpha) - \gamma\}\ D_1 \pm \{\ \beta^2(\delta(1+\alpha)-\gamma)^2\ D_1^2$$
$$- 2(\beta+\beta^2)\ ((\delta(1+\alpha)-\gamma)^2\ D_1^2 - \nu(1+\alpha)D_1)\}^{1/2}]/(\beta+\beta^2)$$
$$(2.48)$$

The solution of the model exists only for positive values of Q_1 and Q_2.

2.5 Numerical Example:

Consider two-item inventory system where in stock out condition of one, the other can be used to substitute its demand. We shall be considering an example, the input data for this example is given in Table 2.1.

For obtaining the optimal solution we use the following steps.

1. Since $h_1 < h_2$,

$S_1 = 0$ and the procedure given in Sec. 2.4.1 is followed.

Values of E, F, G are calculated from Eq. (2.31) and are given as follows:

$$E = .30125, \quad F = 580.68181 \text{ and } G = -91683.884$$

Since $F^2 > 4EG$, therefore both values of S_2 are real,

$$S_2 = 146.72196,$$
$$S_2 = -2074.2964$$

We calculate the decision variables (Q_1, Q_2) corresponding to $S_2 = 146.72196$ and $S_2 = 0$ (following Case III)

The total cost corresponding to $S_2 > 0$ is Rs.131656 and total cost corresponding to $S_2 = 0$ is Rs. 131574. Therefore optimal stocking policy with substitution for the model is corresponding to $S_2 = 0$. The results for the optimal solution for the substitution model and EOQ model are shown in Table 2.2.

Table 2.1: Input data for the numercal example

Input Parameters	Item 1	Item 2
1. Annual demand	1000.0	2000.0
2. Fixed cost of ordering per cycle.	400.0	600.0
3. Holding cost (Per unit / Time)	8.8	11.0
4. Substitution cost / unit.	1.5	1.0
5. Unit cost of item.	42.0	41.0

Table 2.2 : Solution for numerical example.

Model	Item	Procurement Quant/cycle	Subst Quant	Ordering Cost	Purchase Cost	Holding Cost	Subst. Cost	Total cost	Redu-tion
EOQ.	1	301.5	-	3896	124000	3896		131791	-
	2	467.1	-						
Case 1	1	265	167.	3275	123869	3277	1235	131656	136
	2	651	127.						
Case 3	1	406	214.	2860	124161	2860	1693	131574	218
	2	643	271.						

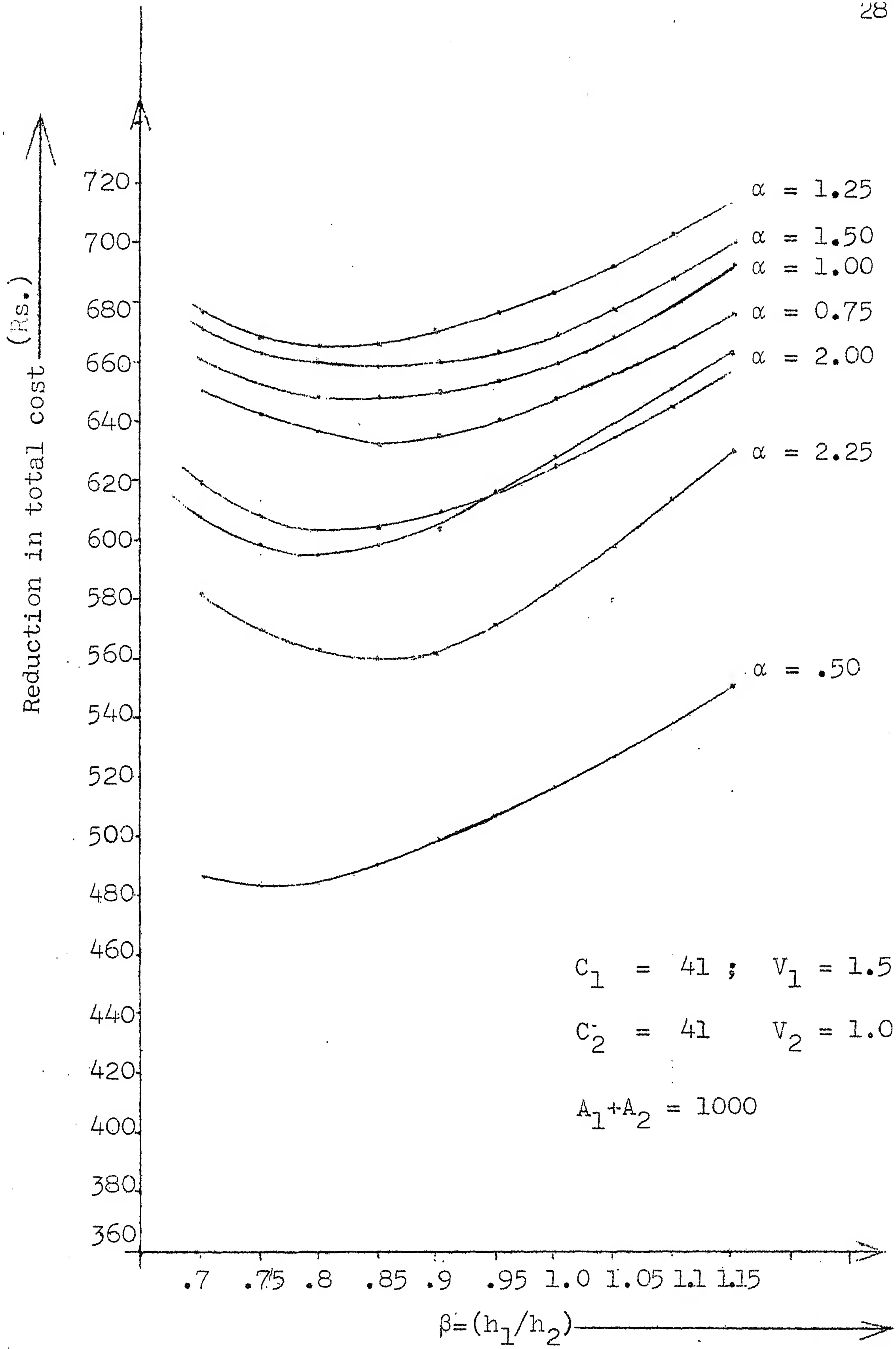

Fig. 2.5(a): Effect of α,β on reduction in total cost.

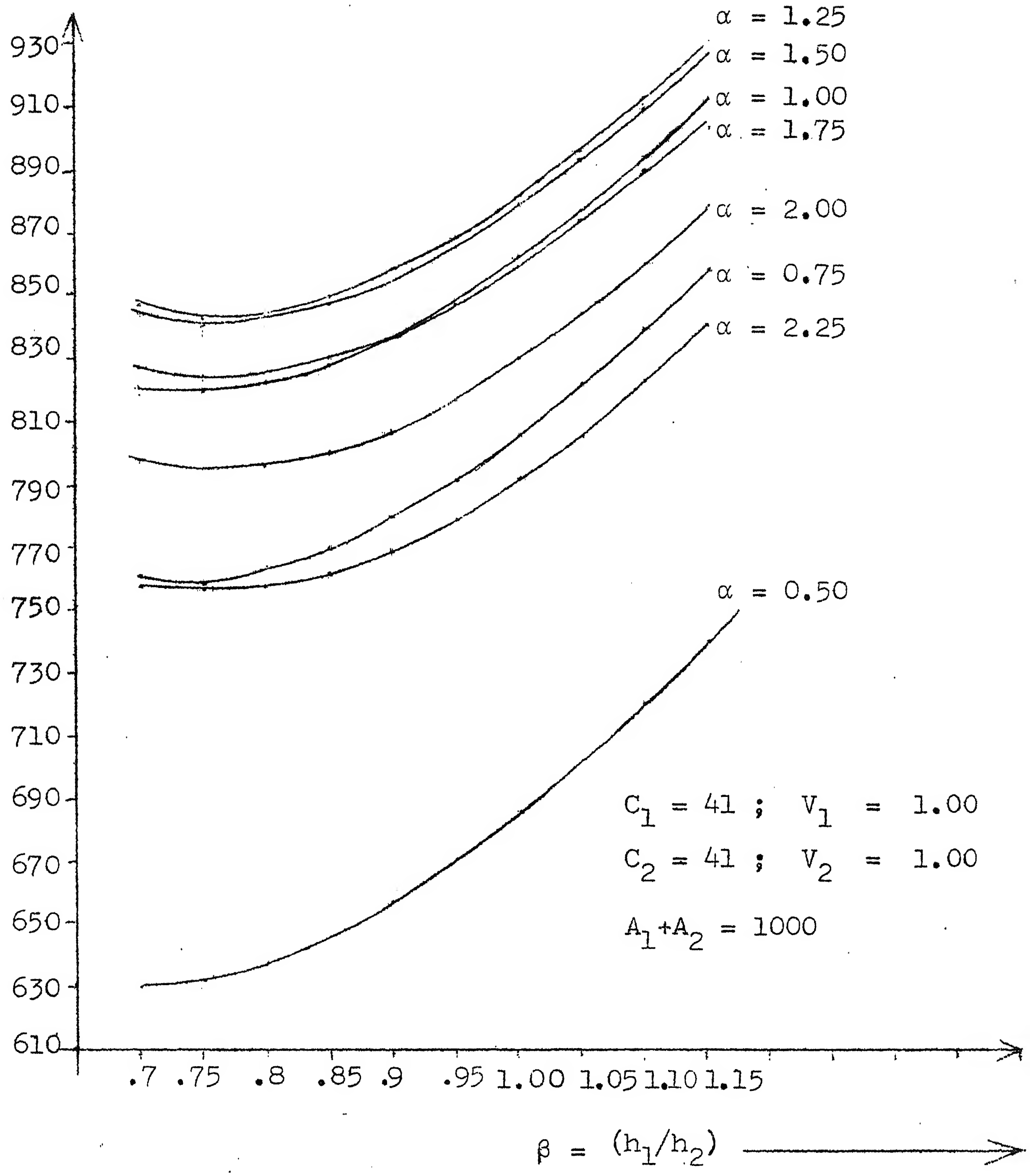

Fig. 2.5(b): Effect of α,β on reduction in total cost.

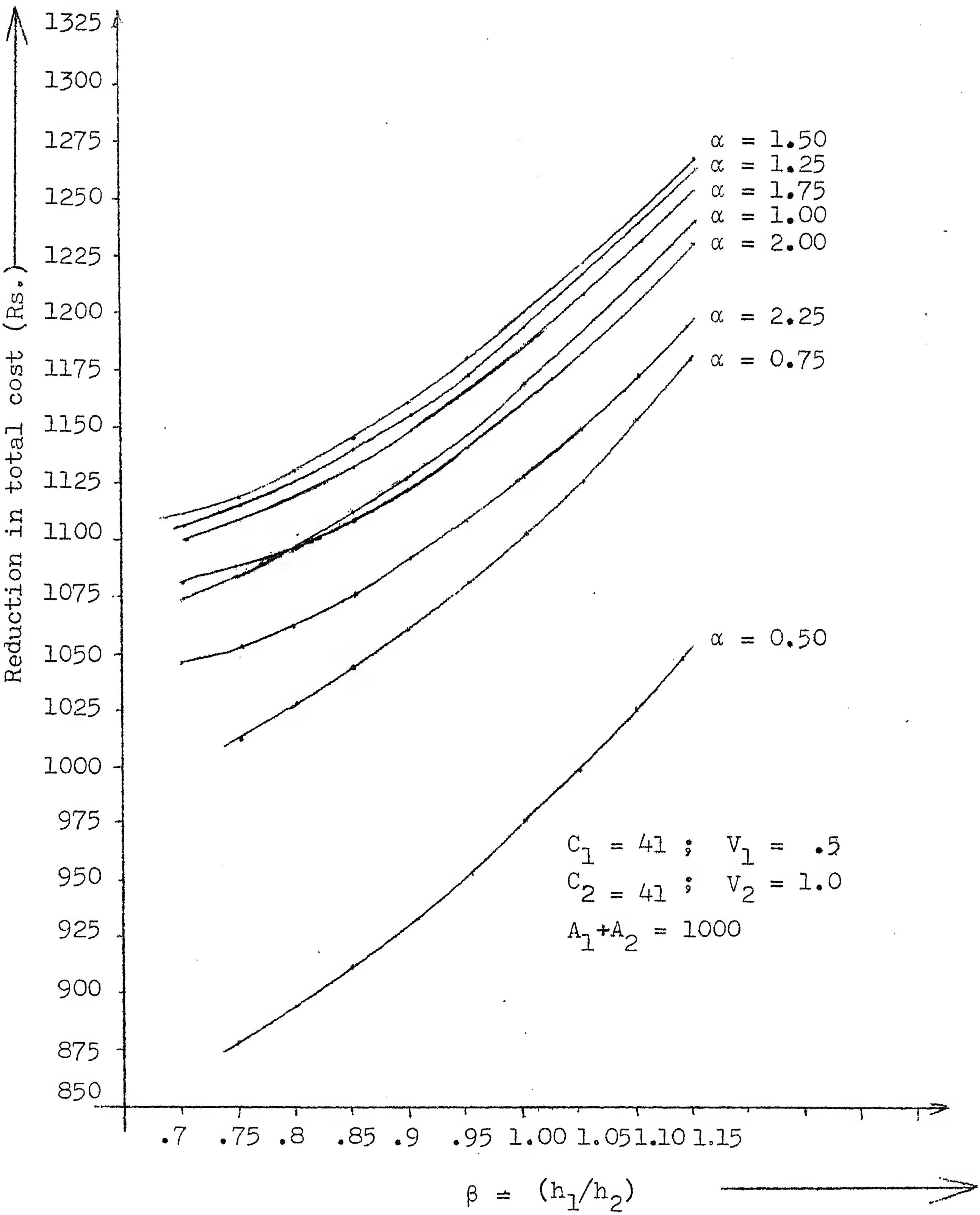

Fig. 2.5(c): Effect of α,β on reduction in total cost.

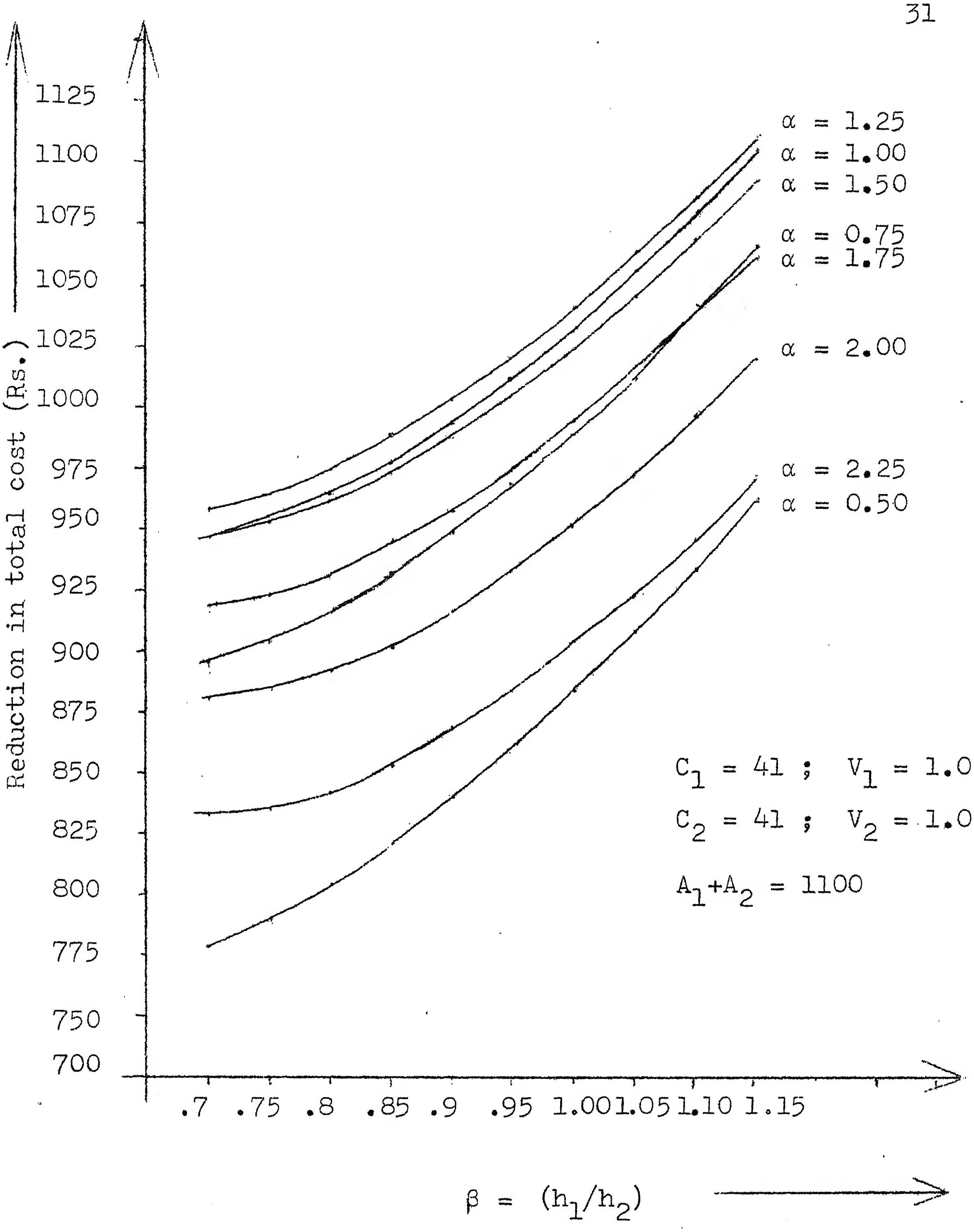

Fig. 2.5(d): Effect of α, β on reduction in total cost.

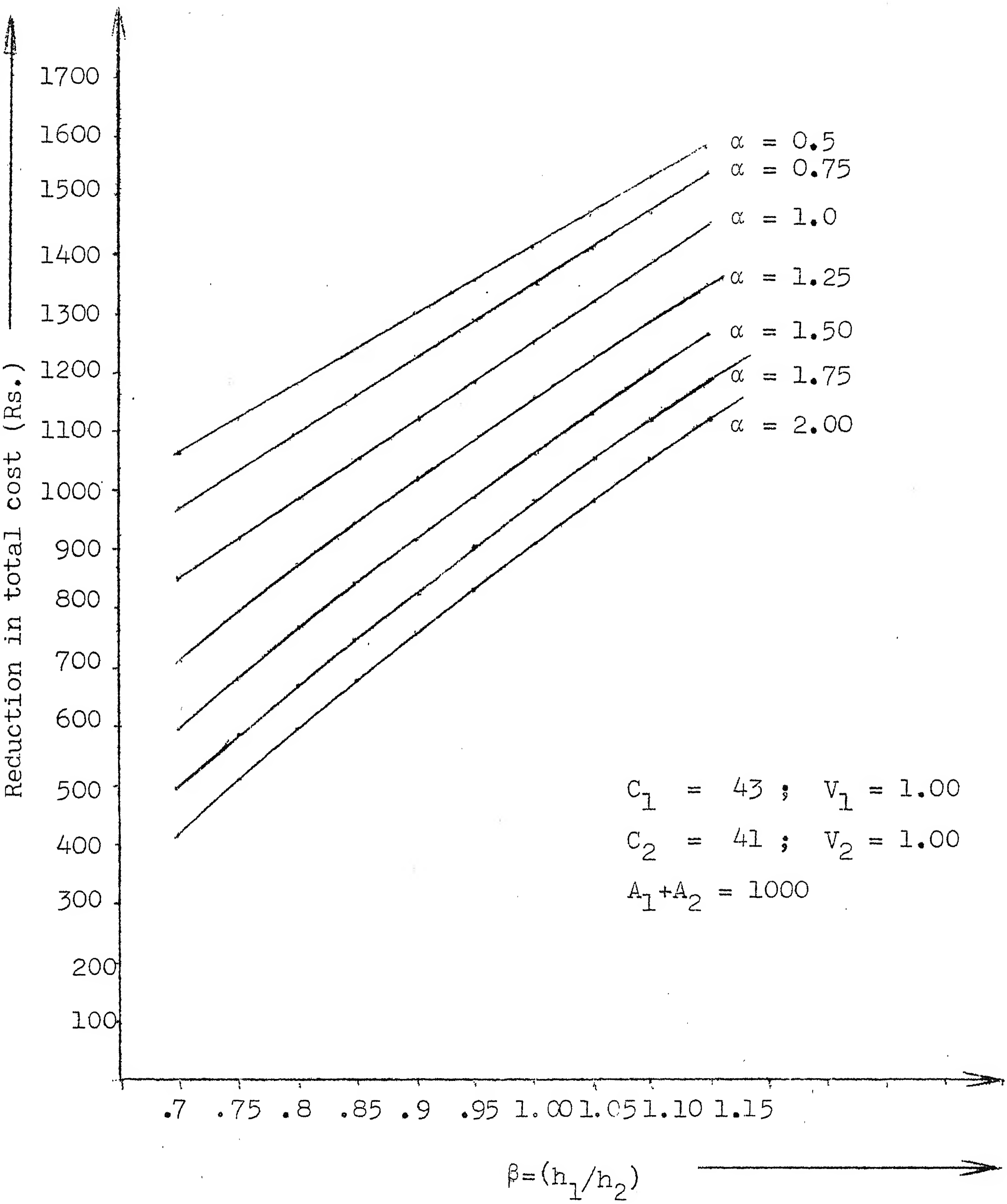

Fig. 2.5(e): Effect of α,β on reduction in total cost.

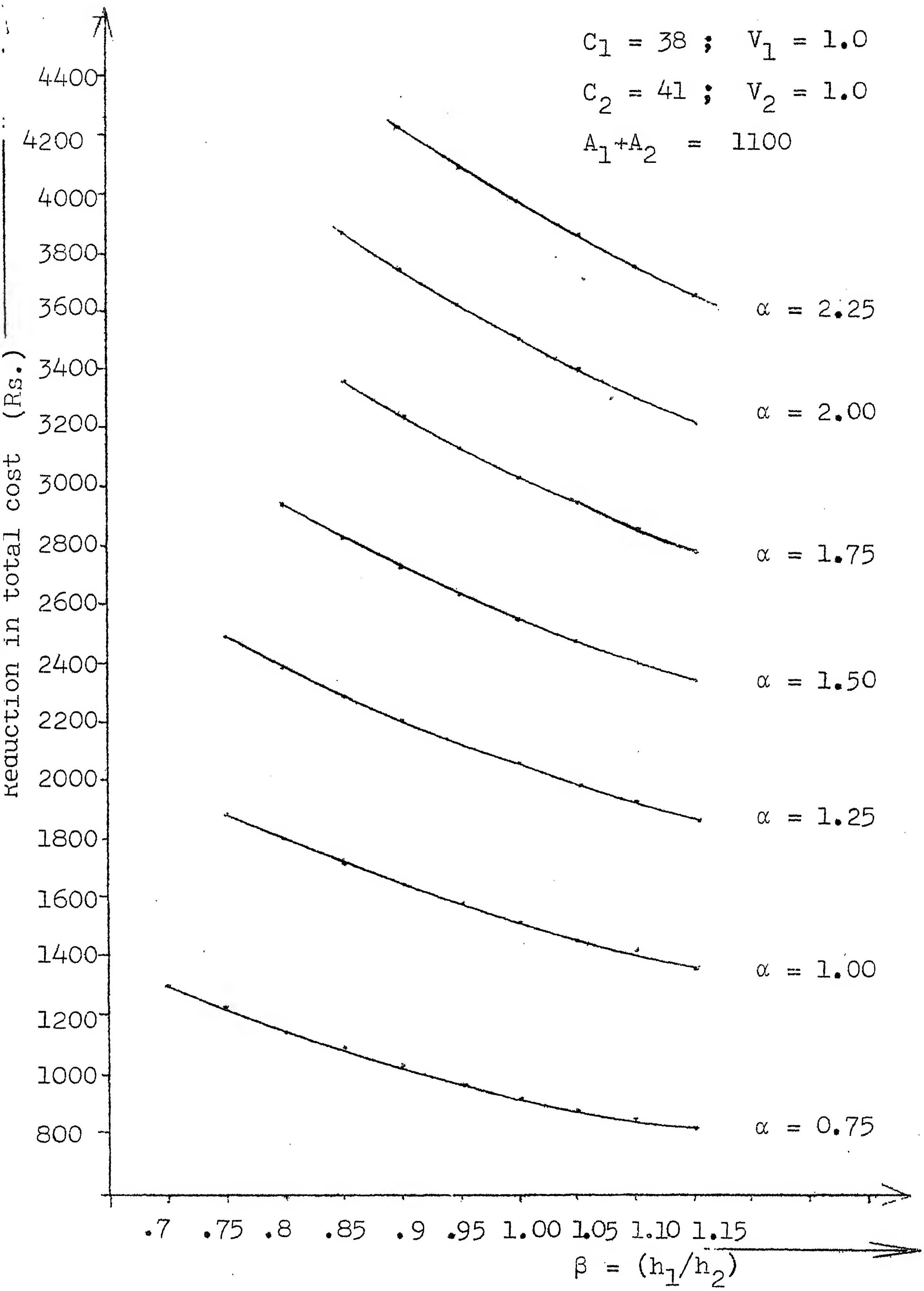

Fig. 2.5(f): Effect of α, β on reduction in total cost.

Comparing various cost-components corresponding to substitution model with EOQ model, we find that annual holding and fixed ordering costs for the former case are less than that of the later. Apart, from this, the stocking policies for the substitution is quite different from EOQ model. In substitution model, item with lesser unit cost is procured more.

2.6 Parametric Analysis:

It has been shown in the previous sections that the total cost per unit time, as obtained by using EOQ, can be reduced considerably if the items are allowed to substitute each other. In this section we are focussing our attention on the variations of this reduction in total cost because of the variations of the various input parameters. These input parameters for various items are:

(1) Unit holding cost

(2) Unit item cost

(3) Unit substitution of cost

(4) Fixed ordering cost

(5) Annual demand

Since we are considering a two item inventory system, we will fix all the cost parameters of one item and demand of the other item and vary the cost parameters of the other item alongwith the demand of first one. The effect of

holding cost, fixed cost, substitution cost and demand on the variation of total annual cost depends on the relation between C_1 and C_2 following three conditions are possible:

1. $C_1 = C_2$:

(a) Let the substitution cost of item 1 be more than the substitution cost of item 2. For this case with any increase in holding cost of item one, the reduction in total cost first increases and then decreases. On the other hand with any increase in the demand of item two, reduction in total cost first increases and then decreases. The value of h_1, which corresponds to the minimum reduction in total cost, also varies with variations in demand. This variation in the reduction of total cost with respect to h_1 and D_2 is shown in Fig. 2.5(a).

From Fig. 2.5(a), (b) and (c), it can be seen that any decrease in the value of unit substitution cost increases the reduction in total cost.

From Fig. 2.5(d) we see that with any increase in fixed ordering cost, the reduction in the total cost is increasing but the nature of the reduction in total cost with respect to h_1 and D_2 remains unchanged.

Explanation:

Initially holding cost of item 1 is lower than that of item 2. As a result of this, more of item 1 is

procured to substitute the demand of item 2. Any increase in h_1 will, therefore, cause much more increase in total cost for the substitution model then the increase in total cost for EOQ model. This tantamounts to saying that increase in h_1 results in a decrease in the reduction of total cost. If h_1 is further increased, then there will be a tendency for procuring more of item 2 and less of item 1. As a result of this any further increase in h_1 will have less effect on the total cost for substitution model than the effect on the total cost for EOQ Model. This finally leads to saying that any increase in h_1, now, causes increase in reduction of total cost per unit time.

As the demand of item 2 increases there is a tendency to procure more and more of item 1 (because $h_1 < h_2$). The increase in holding cost of items in substitution model is very much less than the increase in holding of items EOQ model. As a result of this the reduction in total annual cost increases with increase in D_2. Further increase in D_2 results in higher procurement of item 1 and therefore shorter cycle time. This reduction in cycle time for both the items causes the fixed ordering cost for both the items to go up at a much faster rate than the corresponding increase in fixed ordering cost of EOQ model. Besides this ordering cost, the substitution cost also goes up with increase in D_2. When this increase in substitution and fixed

ordering cost over comes the effect of gain in holding cost than the reduction in total annual cost, keeps decreasing, with any further increase in D_2.

Effect of decrease in substitution cost is to encourage more and more substitution because of the additional saving. As a result of this, additional reduction in total annual cost, increases with decrease in substitution cost.

There is a saving in fixed ordering cost because of less number of annual orders for substitution model as compared to the EOQ model. Any increase in fixed ordering will increase the total cost for EOQ model at much faster rate than that for the substitution model. This finally amounts to some additional saving for substitution model.

(2) $C_1 > C_2$:

For the ease when unit item cost for item 1 is much higher as compared to C_2, and the holding cost of item one is lower than that of item 2 then the effect of increase in h_1 and D_2, as observed from the Fig. 2.5(e), can be stated as:

(a) Any increase in h_1 increases the reduction in total cost.

(b) Any increase in D_2 decreases the reduction in total cost.

Explanations:

Since $C_2 < C_1$, the general tendency in substitution model will be to substitute the demand of item 1 by item 2 and as a result of this item 2 is procured much more than

its demand and item 1 is less than its demand. As h_1 increases, more and more of item two is procured which results in more and more increase in reduction in total cost.

As the demand for item 2 goes up, the total procured quantity, of item 2 per unit time, also goes up this finally results in shorter time periods for both the items. As the number of cycles per year for both the items increases the related increase in total cost for the substitution model is much faster than the corresponding increase is total cost in EOQ model which is only due to increase in number of cycles for item 2 alone. This finally results in decrease in reduction in total annual cost.

(3) $C_2 > C_1$:

For the case when unit item cost for item 1 is much lower than that for the item 2 and the holding cost of item 1 is lower than that of item 2 then the effect of any increase in h_1 and D_2, as observed from Fig. 2.5(f) can be of the following form.

(a) With increase in h_1 the reduction in total cost decreases.

(b) With increase in D_2 the reduction in total cost increases.

Explanations:

For $C_2 > C_1$ and $h_2 > h_1$ it will always be preferred to procure more of item 1 and less of item 2 so that item 1 can

substitute the demand for item 2. The saving in total cost due to substitution is primarily due to both, less item cost, and less holding cost.

As the holding cost for item 1 increases, the saving in holding cost, as mentioned above will start decreasing and as a result, the reduction in total annual cost will decrease. On the other hand, as the cost for item 2 increases the corresponding saving by substituting item 1 by item 2 also increases and therefore the reduction in total cost increases.

<u>General Observation</u>:

1. With any increase in unit substitution cost of an item, the reduction in total annual cost decreases. But the nature of curve of reduction in total cost with respect to holding cost and demand remains same.

2. With any increase in fixed cost, of one item, the reduction in total annual cost increases. But nature of the curve of reduction in total annual cost remains same.

3. As $|C_1-C_2|$ is large, the reduction in total annual cost is more. More number of units of the item with lesser cost is procured to satisfy the demand of the other item. The nature of curve of reduction in total annual cost is dependent on the difference in unit cost of items, as it was discussed earlier in this section.

The conclusion that we have drawn by fixing the cost-parameters of item two and the demand of item one will still hold good if we fix the cost parameters of item one and the demand of item two.

CHAPTER III

SUBSTITUTION OF TWO ITEMS WITH PARTIAL BACKLOGGING

3.1 Problem Statement:

The model developed in Chapter II deals with complete substitution of one item with another in stock-out conditions. In this chapter, we relax the condition of complete substitution and shall discuss the case where, in a stockout condition, an item is partially substituted and the balance is backlogged. The system described is representative of the situation where, owing to the special relative values of cost parameters and restricted procurement availabilities, substitution and backlogging both become necessary. Also, in case of retailing, such mixed substitution and backlogging may occur when some customers want to purchase specific type of brand, quality, style and price valued items and they can wait till that item is in stock position. While other customers are willing to purchase an-other item with similar characteristics.

For the system described above we shall find out the optimal stocking policy such that the total cost per unit time of the system is a minimum. We shall consider the case where the inventory point stocks two items.

For the development of the model, we retain the assumptions of Sec. 2.2 about the demand rate, replenishment,

planning horizon and unit variable costs. As stated above in stock-out situation, there is partial substitution of the demand and balance is backlogged and the demands of both items are not backlogged simultaneously. The decision variables are thus procurement quantities and the proportion of substitution or equivalently the proportion of backlogging.

In addition to the notations used in Sec. 2.3, the following notations are introduced for the analysis of the present system. For item i, $i = 1, 2$

Parameters:

π_i Back ordered cost for item i charged Rs/unit of shortage.

$\tilde{\pi}_i$ The backordered cost for item i in Rs/unit-shortage/time.

Decision Variable:

k_i The proportion of backlogged demand of item i

b_i The number of units of item i backlogged in a cycle.

3.2 Formulation:

Fig. 3.1 depicts the inventory model of the two items with partial substitution. Q_1 units of item 1 are procured at the time when item 2 is completely exhausted. The analysis of the model described in Sec. 2.4 does not

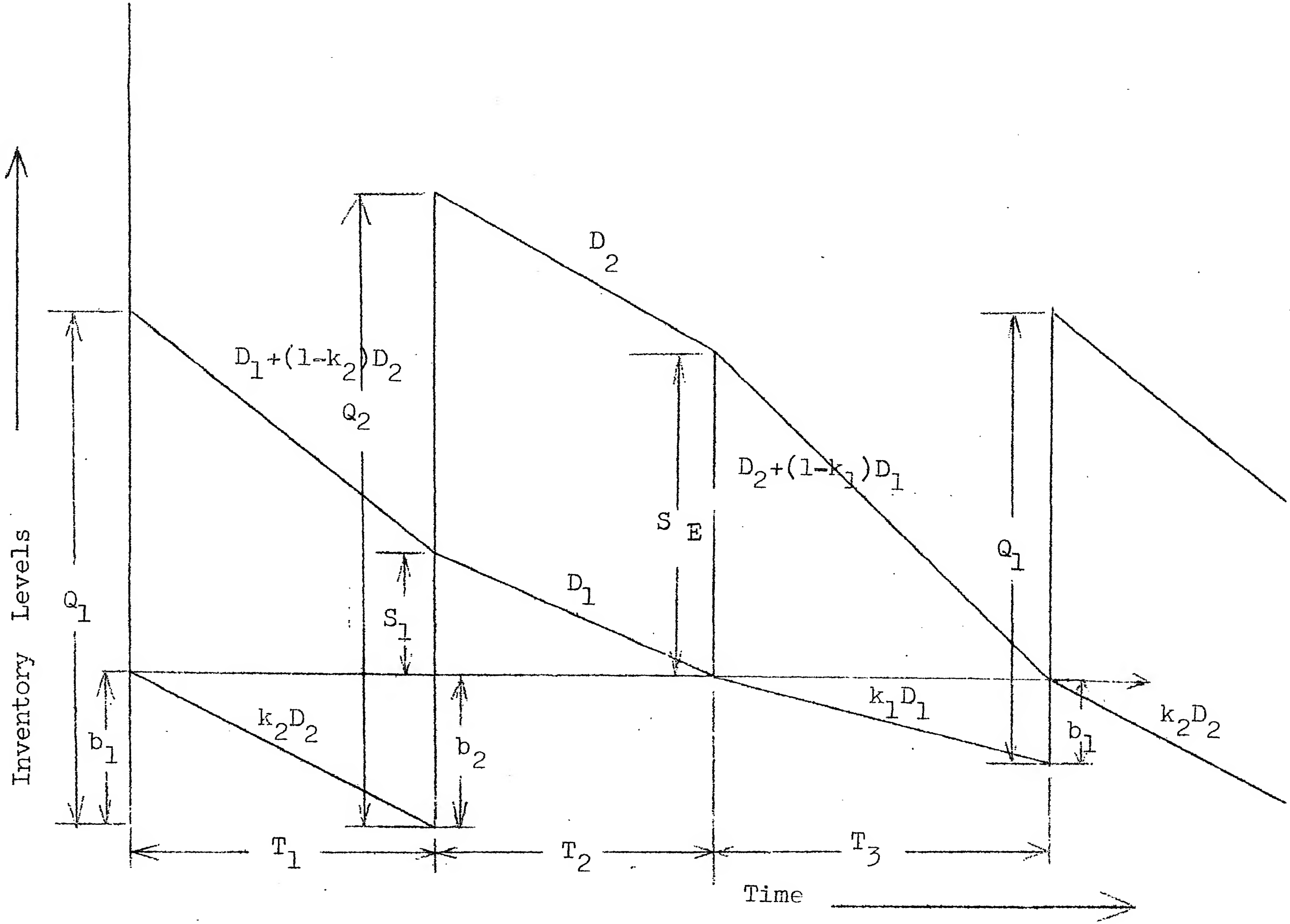

Fig. 3.1: Substitution with partial backlogging.

reduce the total cost that using from EOQ. With this in view, we consider here the case which possibly could reduce the cost. At the beginning of cycle Q_1 units of item 1 are replenishment from which b_1 units of item 1 backordered in previous cycle are satisfied. Thus, on hand inventory is (Q_1-b_1) units. For the duration T_1, item 1 satisfies its own demand and the partial demand of item 2, thus depletes at the rate $[D_1 + (1-k_2) D_2]$ while the demand of item 2 is backlogged at the rate, k_2D_2. At the end of T_1, S_1 units of item 1 are left and Q_2 units of item 2 are replenished from which b_2 backlogged units of item 2 are satisfied. Thus on hand inventory of item 2 is (Q_2-b_2) units. S_1 units of item 1 deplete with the rate D_1 in time T_2. At the end of T_2, the inventory level of item 2 is S_E. The next procurement of item 1 is after time T_3. Therefore, item 2 depletes at the rate, $[D_2+(1-k_1)D_1]$, satisfying its own demand and the partial demand of item 1 until item 2 is exhausted. At the end of T_3, b_1 units of item 1 are backlogged, and now item 1 is procured and thus cycle repeats.

From the figure the following relation can easily be derived.

$$T_1 = \frac{b_2}{k_2D_2} = \frac{(Q_1-b_1) - S_1}{D_1+(1-k_2)D_2} \qquad (3.1)$$

$$T_2 = \frac{S_1}{D_1} = \frac{(Q_2-b_2) - S_E}{D_2} \qquad (3.2)$$

$$T_3 = \frac{S_E}{D_2+(1-k_1)D_1} = \frac{b_1}{k_1 D_1} \tag{3.3}$$

and
$$S_1 = (Q_1-b_1) - [D_1+(1-k_2)\ D_2]\ T_1 \tag{3.4}$$

From Eqs. (3.1)-(3.3), we get,

$$k_2 = \frac{b_2(D_1+D_2)}{D_2(Q_1-b_1-S_1+b_2)} \tag{3.5}$$

$$S_E = Q_2-b_2 - \frac{D_2}{D_1} S_1 \tag{3.6}$$

$$k_1 = \frac{b_1(D_1+D_2)}{D_1\ [Q_2-b_2-\frac{D_2}{D_1} S_1 + b_1]} \tag{3.7}$$

Thus,
$$T = \frac{(Q_1+Q_2)}{(D_1+D_2)} \tag{3.8}$$

Various relevant costs per cycle are given as follows:

(A) Procurement Cost:

$$\text{For item 1} = A_1 + C_1\ Q_1 \tag{3.9}$$

$$\text{For item 2} = A_2 + C_2\ Q_2 \tag{3.10}$$

(B) Holding Cost:

$$\text{For item 1} = h_1\ (Q_1^2 - 2Q_1 b_1 + b_1{}^2 + \frac{D_2}{D_1} S_1{}^2 + b_2 S_1)/(2(D_1+D_2)) \tag{3.11}$$

$$\text{For item 2} = h_2(2Q_2S_1 - b_2S_1 - \frac{D_2}{D_1} S_1{}^2 + Q_2{}^2 - 2Q_2b_2 + b_2{}^2 + Q_2b_1 - b_2b_1 - \frac{D_2}{D_1} b_1S_1)/(2(D_1+D_2)) \tag{3.12}$$

(C) Backlogging Cost:

$$\text{For item 1} = \pi_1 b_1 + \frac{\tilde{\pi}_1}{2} b_1 \frac{(Q_2 - b_2 - \frac{D_2}{D_1} S_1 + b_1)}{(D_1 + D_2)} \tag{3.13}$$

$$\text{For item 2} = \pi_2 b_2 + \frac{\tilde{\pi}_2}{2} b_2 \frac{(Q_1 - b_1 - S_1 + b_2)}{(D_1 + D_2)} \tag{3.14}$$

(D) Substitution Cost:

$$\text{For item 1} = V_1(D_1Q_2 - D_1b_2 - D_2S_1 - b_1D_2)/(D_1 + D_2) \tag{3.15}$$

$$\text{For item 2} = V_2(D_2Q_1 - D_2b_1 - D_2S_1 - b_2D_1)/(D_1 + D_2) \tag{3.16}$$

Therefore, total annual cost is given by,

$$\begin{aligned} TC = \frac{1}{(Q_1+Q_2)} & [(A_1+A_2)(D_1+D_2) + \{C_1(D_1+D_2) + V_2D_2\}Q_1 \\ & + \{C_2(D_1+D_2)+V_1D_1\}Q_2 + \frac{1}{2} h_1Q_1{}^2 + \frac{1}{2} h_2Q_2{}^2 \\ & + b_1{}^2 \frac{(h_1+\tilde{\pi}_1)}{2} + b_2{}^2 \frac{(h_2+\tilde{\pi}_2)}{2} + \frac{D_2}{D_1} S_1^2 \frac{(h_1-h_2)}{2} \\ & - h_1Q_1b_1 - h_2Q_2b_2 + \frac{\tilde{\pi}_2}{2} Q_1b_2 + Q_2b_1 \frac{(h_2+\tilde{\pi}_1)}{2} \\ & + h_2Q_2S_1 - \frac{\tilde{\pi}_1}{2} \frac{D_2}{D_1} S_1b_1 + \frac{(h_1-(h_2+\tilde{\pi}_2))}{2} b_2S_1 \\ & - b_2b_1 \frac{(h_2+\tilde{\pi}_1+\tilde{\pi}_2)}{2} + \{\pi_1(D_1+D_2)-(V_1+V_2)\ D_2\}\ b_1 \\ & + \{\pi_2(D_1+D_2)-(V_1+V_2)D_1\}\ b_2 - (V_1+V_2)D_2S_1] \end{aligned} \tag{3.17}$$

Differentiating Eq. (3.17) with respect to the decision variables Q_1, Q_2, b_1, b_2 and S_1 and setting them equal to zero, we get,

$$\frac{\partial TC}{\partial Q_1} = \frac{1}{(Q_1+Q_2)} \Big[C_1(D_1+D_2)+V_2D_2 + h_1Q_1 - h_1b_1 + \frac{\tilde{\pi}_2 b_2}{2} \Big] - \frac{TC}{(Q_1+Q_2)^2} = 0 \qquad (3.18)$$

$$\frac{\partial TC}{\partial Q_2} = \frac{1}{(Q_1+Q_2)} \Big[C_2(D_1+D_2)+V_1D_1+h_2Q_2-h_2b_2 + b_1 \frac{(\tilde{\pi}_1+b_2)}{2} + h_2S_1 \Big] - \frac{TC}{(Q_1+Q_2)^2} = 0 \qquad (3.19)$$

$$\frac{\partial TC}{\partial b_1} = \frac{1}{(Q_1+Q_2)} \Big[(h_1+\tilde{\pi}_1)b_1-h_1Q_1 - \frac{\tilde{\pi}_1}{2} \frac{D_2}{D_1} S_1 - b_2 \frac{(h_2+\tilde{\pi}_1+\tilde{\pi}_2)}{2} + \pi_1(D_1+D_2)-(V_1+V_2)D_2 + Q_2 \frac{(h_2+\tilde{\pi}_1)}{2} \Big] = 0 \qquad (3.20)$$

$$\frac{\partial TC}{\partial b_2} = \frac{1}{(Q_1+Q_2)} \Big[(h_2+\tilde{\pi}_2)b_2 - h_2Q_2 + \frac{\tilde{\pi}_2}{2} Q_1 + \Big\{ \frac{h_1-(h_2+\tilde{\pi}_2)}{2} \Big\} S_1 - \frac{(h_2+\tilde{\pi}_1+\tilde{\pi}_2)}{2} b_1 + \pi_2(D_1+D_2) - (V_1+V_2)\, D_1 \Big] = 0 \qquad (3.21)$$

$$\frac{\partial TC}{\partial S_1} = \frac{1}{(Q_1+Q_2)} \Big[\frac{D_2}{D_1} (h_1-h_2)S_1 + h_2Q_2 - \frac{\tilde{\pi}_1 D_2}{2D_1} b_1 + \frac{h_1-(h_2+\tilde{\pi}_2)}{2} b_2 - (V_1+V_2)D_2 \Big] = 0 \qquad (3.22)$$

Solving Eqs. (3.18) - (3.22), we can express the optimal values of Q_2, b_1, b_2 and S_1 in terms of Q_1. These are given as follows:

$$Q_2 = -K_{12} Q_1 + K_{13} \tag{3.23}$$

$$b_1 = K_{14} Q_1 - K_{15} \tag{3.24}$$

$$b_2 = -K_{16} Q_1 + K_{17} \tag{3.25}$$

and

$$S_1 = -K_{18} Q_1 + K_{19} \tag{3.26}$$

where,

$$K_1 = h_1 + \pi_1 + \frac{\tilde{\pi}_2}{2} \frac{D_2}{D_1} \cdot \frac{1}{h_2} (h_1 + \frac{\tilde{\pi}_1 + h_2}{2})$$

$$K_2 = \frac{h_1}{K_1} (1 + \frac{\tilde{\pi}_2}{2} \frac{D_2}{D_1} \frac{1}{h_2})$$

$$K_3 = \frac{1}{2K_1} (h_2 + \tilde{\pi}_1 + \tilde{\pi}_2 \frac{D_2}{D_1})$$

$$K_4 = \frac{1}{K_1} (\frac{h_2 + \tilde{\pi}_1 + \tilde{\pi}_2}{2} + (\frac{\tilde{\pi}_2}{2} + h_2) \frac{\tilde{\pi}_2}{2} \frac{D_2}{D_1} \cdot \frac{1}{h_2})$$

$$K_5 = \frac{1}{K_1} [(C_1 - C_2 + V_2)(D_1 + D_2) + (1 - \frac{\tilde{\pi}_2}{2h_2}) (V_1 + V_2) D_2 - \pi_1 (D_1 + D_2)]$$

$$K_6 = h_2 + \tilde{\pi}_2 + \frac{1}{h_2} \frac{(h_1 - h_2 - \tilde{\pi}_2)}{2} (h_2 + \frac{\tilde{\pi}_2}{2}) + K_4 \, KM1$$

$$K_7 = \frac{1}{K_6} [\frac{\tilde{\pi}_2}{2} + \frac{h_1}{2h_2} (h_1 - h_2 - \tilde{\pi}_2) - K_2 KM1]$$

$$K_8 = \frac{1}{K_6}\left[h_2 + \frac{h_1 - h_2 - \tilde{\pi}_2}{2} - K_3 . KM1\right]$$

$$K_9 = \frac{1}{K_6}\left[K_5\ KM1 - \pi_2\ (D_1 + D_2) + (V_1 + V_2)\ D_1\right.$$

$$- \frac{(h_1 - h_2 - \tilde{\pi}_2)}{2h_2}\ \{(C_1 - C_2 + V_2)\ (D_1 + D_2)$$

$$\left. - (V_1 + V_2)\ D_1\}\right]$$

$$KM1 = \frac{h_2 + \tilde{\pi}_1 + \tilde{\pi}_2}{2} + \frac{h_1 - h_2 - \tilde{\pi}_2}{2h_2}\ (h_1 + \frac{h_2 + \tilde{\pi}_2}{2})$$

$$K_{10} = [h_2 + (h_2 - h_1)\ \frac{D_2}{D_1} + K_3 . Ka_1 + K_8 . Ka_2]$$

$$K_{11} = \{\ \frac{h_1 - h_2}{h_2}\ h_1 - K_2 . Ka_1\ \}\ \frac{D_2}{D_1} - K_7 . Ka_2$$

$$K_{12} = K_{11}/K_{10}$$

$$K_{13} = \frac{1}{K_{10}}\left[K_5 . Ka_1 + (V_1 + V_2)\ D_2 - K_9 . Ka_2\right.$$

$$\left. - \frac{D_2}{D_1}\ \frac{(h_1 - h_2)}{h_2}\ [(C_1 - C_2 + V_2)(D_1 + D_2) - (V_1 + V_2) D_1]\right]$$

$$Ka_1 = \frac{\tilde{\pi}_1}{2}\ \frac{D_2}{D_1} + (h_1 + \frac{h_2 + \tilde{\pi}_1}{2})\ \frac{(h_1 - h_2)}{h_2}\ \frac{D_2}{D_1}$$

$$Ka_2 = \frac{h_1 - h_2 - \tilde{\pi}_2}{2} + (h_2 + \frac{\tilde{\pi}_2}{2})\ \frac{D_2}{D_1}\ \frac{(h_1 - h_2)}{h_2} + K_4\ Ka_1$$

$$K_{14} = K_2 + K_4(K_7 + K_8\ K_{12}) + K_3\ K_{12}$$

$$K_{15} = K_3\ K_{13} + K_4(K_9 + K_8\ K_{13}) + K_5$$

$$K_{16} = K_7 + K_8 K_{12}$$

$$K_{17} = K_8 K_{13} + K_9$$

$$K_{18} = \frac{h_1}{h_2} + K_{12} - \frac{(h_2 + \tilde{\pi}_2/2)}{h_2} K_{16}$$

$$- \frac{1}{h_2} (h_1 + \frac{\tilde{\pi}_2 + h_2}{2}) K_{14}$$

$$K_{19} = \frac{(C_1 - C_2 + V_2)(D_1 + D_2) - (V_1 + V_2) D_1}{h_2} - K_{13}$$

$$+ \frac{1}{h_2} (h_2 + \frac{\tilde{\pi}_2}{2}) K_{17} + \frac{1}{h_2} (h_1 + \frac{\tilde{\pi}_1 + h_2}{2}) K_{15}$$

Using Eqs. (3.23) - (3.26) in Eq. (3.17), we get optimal value of Q_1 as follows:

$$AQ_1^2 + BQ_1 + C = 0 \tag{3.27}$$

where

$$A = \frac{1}{2} h_1 - h_1 K_{12} - (h_1 + \frac{h_2 + \tilde{\pi}_1}{2}) K_{14} - (h_2 + \frac{\tilde{\pi}_2}{2}) K_{16}$$

$$- \frac{1}{2} h_2 K_{12}^2 - \frac{(h_1 + \tilde{\pi}_1)}{2} K_{14}^2 - \frac{(h_2 + \tilde{\pi}_2)}{2} K_{16}^2$$

$$- \frac{D_2}{2D_1} (h_1 - h_2) K_{18} + h_2 K_{12} - \frac{\tilde{\pi}_1}{2} \frac{D_2}{D_1} K_{14} K_{18}$$

$$+ \frac{h_1 - (h_2 + \tilde{\pi}_2)}{2} K_{16} K_{18} - \frac{h_2 + \tilde{\pi}_1 + \tilde{\pi}_2}{2} K_{14} K_{16}$$

$$B = - \{(C_1 - C_2 + V_2)(D_1 + D_2) - (V_1 + V_2) D_1\} K_{12}$$

$$+ h_1 K_{13} + (h_1 + \frac{h_2 + \tilde{\pi}_1}{2}) K_{15}$$

$$- \left(h_2 + \frac{\tilde{\pi}_2}{2}\right) (K_{12} K_{17} + K_{13} K_{16})$$

$$+ h_2 K_{12} K_{13} + (h_1 + \tilde{\pi}_1) K_{14} K_{15} + (h_2 + \tilde{\pi}_2) K_{16} K_{17}$$

$$+ \frac{D_2}{D_1} (h_1 - h_2) K_{18} K_{19} + h_2 (K_{12} K_{19} + K_{13} K_{18})$$

$$+ \frac{\tilde{\pi}_1}{2} \frac{D_2}{D_1} (K_{15} K_{18} + K_{14} K_{19}) + \frac{h_1 - (h_2 + \tilde{\pi}_2)}{2} \times$$

$$\times (K_{16} K_{19} + K_{15} K_{18}) + \frac{h_2 + \tilde{\pi}_1 + \tilde{\pi}_2}{2} (K_{15} K_{16}$$

$$+ K_{14} K_{17}) - \{\pi_1 (D_1 + D_2) - (V_1 + V_2) D_2\} K_{14}$$

$$+ \{\pi_2 (D_1 + D_2) - (V_1 + V_2) D_1\} K_{16} + (V_1 + V_2) D_2 K_{18}$$

$$C = - (A_1 + A_2)(D_1 + D_2) + \{(C_1 - C_2 + V_2)(D_1 + D_2)$$

$$- (V_1 + V_2) D_1\} K_{13} + \left(h_2 + \frac{\tilde{\pi}_2}{2}\right) K_{13} K_{17}$$

$$- \frac{1}{2} h_2 K_{13}^2 - \frac{(\tilde{\pi}_1 + h_1)}{2} K_{15}^2 - (h_2 + \tilde{\pi}_2) K_{17}^2$$

$$- \frac{D_2}{D_1} (h_1 - h_2) K_{19}^2 - h_2 K_{13} K_{19} - \frac{\tilde{\pi}_1}{2} \frac{D_2}{D_1}$$

$$- \frac{h_1 - (h_2 + \tilde{\pi}_2)}{2} K_{17} K_{19} + \frac{h_2 + \tilde{\pi}_1 + \tilde{\pi}_2}{2} (-K_{15} K_{17})$$

$$+ \{\pi_1 (D_1 + D_2) - (V_1 + V_2) D_2\} K_{15} + \{\pi_2 (D_1 + D_2)$$

$$- (V_1 + V_2) D_1\} K_{17} - (V_1 + V_2) D_2 K_{19}.$$

From Eq. (3.27), the optimal solution of Q_1 is given by,

$$Q_1 = -\frac{B \pm (B^2-4AC)^{1/2}}{2A} \qquad (3.28)$$

The positive root of Q_1 is taken and put in Eqs. (3.23) - (3.26) to obtain the optimal values of Q_2, b_1, b_2 and S_1. If all decision variables have the positive-values, the solution exists for above model otherwise it would indicate that the values of parameters are such that the substitution with partial backlogging is not feasible.

We calculate the total annual cost given by Eq. (3.17). Now we solve the problem when the inventory system of both items are operating independently with back logging. The corresponding total annual cost is given by,

$$TCB = \sum_{i=1}^{2} \left[\frac{A_i D_i}{Q_i} + h_i \frac{(Q_i - b_i)^2}{2Q_i} + \pi_i b_i \frac{D_i}{Q_i} + \frac{\tilde{\pi}_i b_i^2}{2Q_i} + C_i D_i \right] \qquad (3.29)$$

where,

$$Q_i = \sqrt{\frac{2A_i D_i}{h_i} - \frac{(\pi_i D_i)^2}{h_1(h_i+\tilde{\pi}_i)}} \sqrt{\frac{h_i+\tilde{\pi}_i}{\tilde{\pi}_i}} \qquad \text{for } i = 1,2$$

$$b_i = \frac{h_i Q_i - \pi_i D_i}{h_i + \tilde{\pi}_i} \qquad \text{for } i = 1,2$$

If TC < TCB, we prefer substitution with partial backlogging.

CHAPTER IV

JOINT REPLENISHMENT WITH END SUBSTITUTION

4.1 Problem Statement:

The models developed in previous chapters deal with the substitution of one item with another in stock-out conditions when items are procured individually. Now we consider the situation when both items are ordered (and received) simultaneously with the possibility of a reduction in the cost then in stock-out situation demand of an item can be satisfied by the other. We concentrate on the case when only one item can be substituted by other. This one way substituion may be representative of a situation where better quality product can be used as a substitute .

For the system described above, we shall find out the optimal stocking policy such that total annual cost of inventory system is minimum.

We shall analyze the problem for two different cases. In Sec. 4.2, we discuss the problem of complete substitution. Section 4.3 describes the problem of partial substitution.

4.2 Complete Substitution:

For the development of the model we retain the assumptions of Sec. 2.2 of Chapter II about demand rate, replenishment,

planning horizon and unit variable cost. Furthermore, we assume that item 2 is completely substituted by item 1. Thus the decision variables are procurement quantities of two items.

4.2.1 Notations:

Following notations are used for the mathematical development of present model.

For item 1, $i = 1, 2$,

D_i Annual demand rate of item i

h_i Holding cost of item i in Rs./Unit/Unit time.

i_i Inventory carrying charge of item i, in Rs./Rs./Unit time

Q_i Replenishment quantity of item i in units

A Set-up or fixed ordering cost for joint replenishment

V_2 Unit substitution cost of item 2

T Cycle time

4.2.2 Formulation:

The inventory levels of item 1 and 2 are shown in Fig. 4.1. Referring to the figure we see that at the beginning of the cycle, Q_1 units of item 1 and Q_2 units of item 2 are replenished jointly. The Q_2 units of item 2 deplete in time T_1 with the rate D_2. The next procurement of item 2 is after

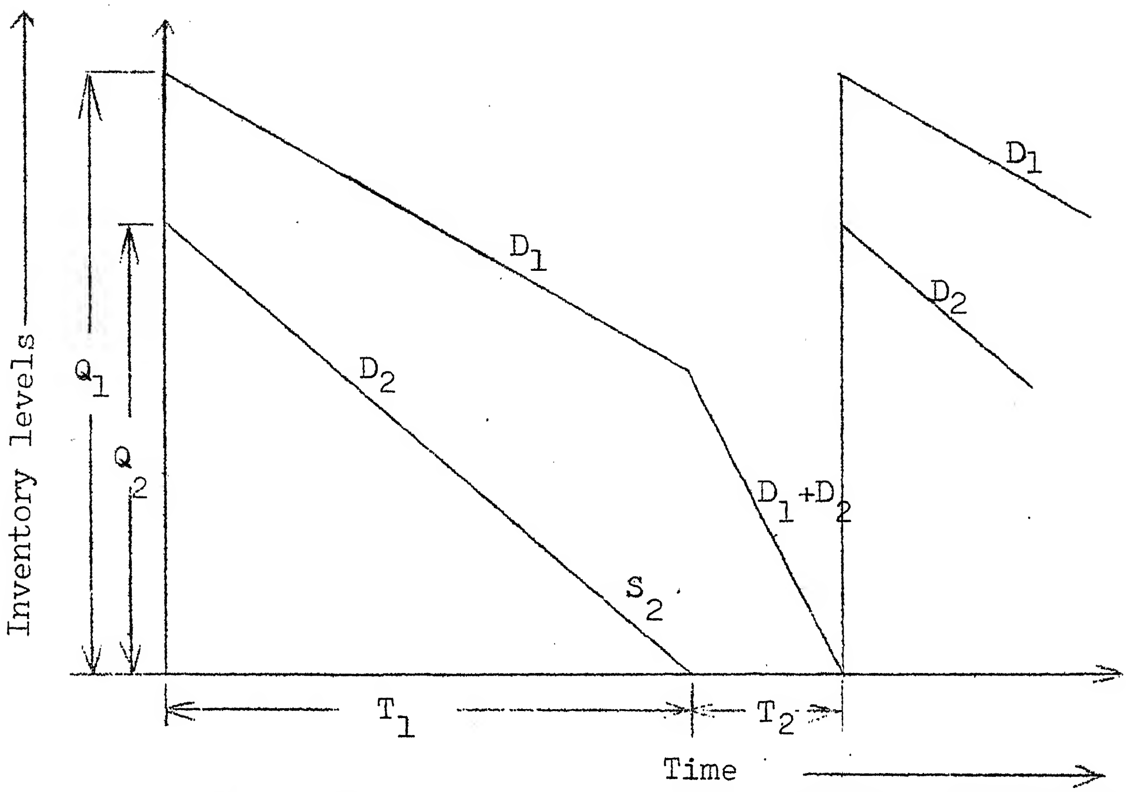

Fig. 4.1: Joint replenishment with complete substitution.

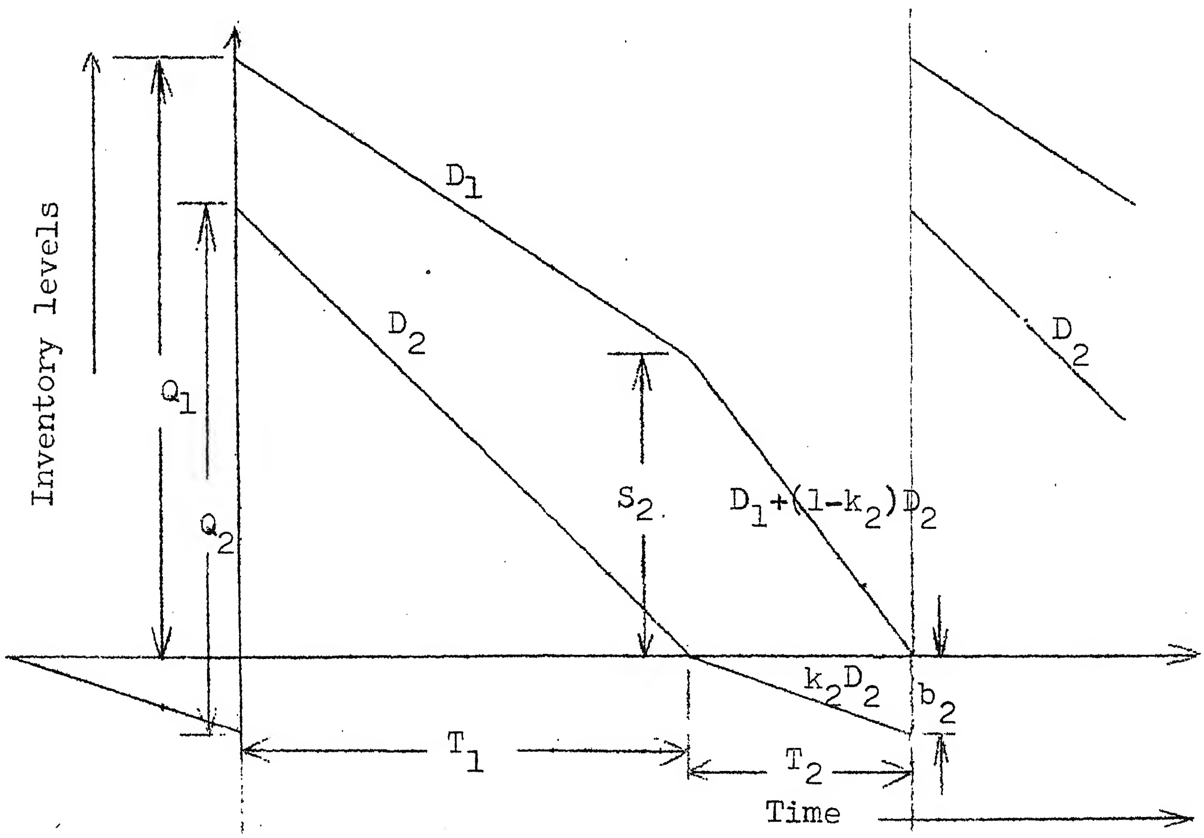

Fig.4.2: Joint replenishment with partial substitution.

time T_2. Thus item 1 substitutes the demand of item 2 for the duration T_2. Therefore, as shown in the figure units of item 1 deplete at the rate of D_1+D_2 during the time T_2. At the end of T_2, item 1 and item 2 are replenished jointly thus cycle repeats.

From the figure following relations can be easily derived.

$$T_1 = Q_2/D_2 \tag{4.1}$$

$$T_2 = S_2/(D_1+D_2) \tag{4.2}$$

$$S_2 = Q_1 - \frac{Q_2}{D_2} D_1 \tag{4.3}$$

Thus cycle time,

$$T = T_1 + T_2$$

$$= \frac{(Q_1+Q_2)}{(D_1+D_2)} \tag{4.4}$$

The various relevant costs per cycle are given as follows:

(A) Procurement Cost $= A + C_1Q_1 + C_2Q_2$ (4.5)

(B) Holding Cost:

For Item 1 $= \dfrac{Q_1Q_2h_1}{(D_1+D_2)} + \dfrac{Q_1{}^2h_1}{2(D_1+D_2)} - \dfrac{D_1}{2D_2}\dfrac{Q_2{}^2h_1}{(D_1+D_2)}$ (4.6)

For Item 2 $= \dfrac{Q_2^2\, h_2}{2D_2}$ (4.7)

(C) Substitution Cost

$$= V_2 \ D_2 \ T_2$$

$$= \frac{V_2 \ D_2}{(D_1+D_2)} \ (Q_1 - \frac{D_1 \ Q_2}{D_2}) \quad (4.8)$$

And the total annual cost is given by,

$$TC = \frac{1}{(Q_1+Q_2)} \left[\frac{h_1 Q_1^2}{2} + h_1 Q_1 Q_2 + \frac{Q_2^2}{2D_2} \{h_2(D_1+D_2) - h_1 D_1\} \right.$$

$$+ A(D_1+D_2) + \{C_1(D_1+D_2) + V_2 D_2\} \ Q_1$$

$$\left. + \{C_2(D_1+D_2) - V_2 D_1\} \ Q_2 \right] \quad (4.9)$$

Setting,

$$\frac{\partial TC}{\partial Q_1} = 0 \quad \text{and} \quad \frac{\partial TC}{\partial Q_2} = 0, \quad \text{we get,}$$

$$\frac{\partial TC}{\partial Q_1} = \frac{1}{(Q_1+Q_2)} [h_1 Q_1 + h_1 Q_2 + C_1(D_1+D_2) + V_2 D_2] -$$

$$- \frac{TC}{(Q_1+Q_2)^2} = 0 \quad (4.10)$$

$$\frac{\partial TC}{\partial Q_2} = \frac{1}{(Q_1+Q_2)} [h_1 Q_1 + \frac{Q_2}{D_2} \{h_2(D_1+D_2) - h_1 D_1\}$$

$$+ C_2(D_1+D_2) - V_2 D_1] - \frac{TC}{(Q_1+Q_2)^2} = 0 \quad (4.11)$$

Solving eqns. (4.10) and (4.11) we get the optimal Q_2 and Q_1

$$Q_2 = k_Q \quad (4.12)$$

where,

$$k_Q = \frac{C_1 - C_2 + V_2}{(h_2 - h_1)}$$

$$Q_1 = -k_Q + \sqrt{[k_Q^2 - 2[\frac{\delta}{\beta}(1+\alpha)\ D_1\ k_Q - k_Q^2}$$

$$\{\frac{1}{\beta}(\frac{1+\alpha}{2\alpha}) - \frac{1}{2\alpha}\} - \frac{\nu}{\beta}(1+\alpha)\ D_1] \qquad (4.13)$$

Only positive values of Q_1 and Q_2 are taken.

For $Q_2 > 0$,

$$\frac{C_1 - C_2 + V_2}{(h_2 - h_1)}\ D_2 > 0$$

We get following cases,

(d) For $h_2 > h_1$, $C_1 - C_2 + V_2 > 0$ $C_1 > (C_2 - V_2)$

if inventory carrying charge is same for both items,

$$iC_2 > iC_1 \quad \text{or } C_2 > C_1$$

or $$(C_2 - V_2) < C_1 < C_2 \qquad (4.14)$$

i.e. for positive value of Q_2, the condition given in Eq.(4.14) must be satisfied.

(e) If $h_2 < h_1$, $C_1 - C_2 + V_2 < 0$ for $Q_2 > 0$, otherwise Q_2 is zero.

(f) If $h_2 = h_1$, substitution is not profitable. The interaction in the demand for substitutable item is uneconomic in joint procurement policy.

When both items are procured jointly and there is no substitution, the total annual cost is given by,

$$TCI = C_1 D_1 + C_2 D_2 + \sqrt{2A(h_1 D_1 + h_2 D_2)} \qquad (4.15)$$

Under joint procurement, the substitution is profitable if,

$$TC < TCI$$

where TC has been obtained from Eq. (4.9).

4.3 Partial Substitution:

Now we describe the inventory system of jointly procured items where owing to special relative values of the cost parameters and restricted procurement as well as customer preferences to purchase an item, in stock-out condition an item is partially substituted and the balance is backlogged. We shall consider the case again where inventory point stocks two similar type of items.

For the development of model, we retain the assumption of Sec. 2.2 of Chapter II about replenishment, demand rate, planning horizon and unit variable costs. Furthermore, in out-of-stock situation, item 1 partially substitutes item 2. The back-logging of the substitute is not allowed. The decision variables are procurement quantities and the proportion of substitution or equivalently the proportion of back-logging.

In addition to the notation used in Sec. (4.2.1), the following notations are introduced for the analysis of the present system.

π_2 : Back order cost for item 2 in Rs./Unit of shortage

π_2'' The back order cost for item 2 in Rs./Unit of shortage/unit time.

k_2 : The proportion of backlogged demand of item 2 $(0 \leq k_2 \leq 1)$

b_2 : The number of units of item 2 backlogged in a cycle.

4.3.1 Formulation:

The inventory levels are shown in Fig. 4.2. At the beginning of the cycle, Q_1 units of item 1 and Q_2 units of item 2 are procured jointly from which b_2 backlogged units of item 2 are satisfied. Thus on hand inventory of item 2 is $(Q_2 - b_2)$ units $(Q_2 - b_2)$units of item 2 depletes in time T_1 with rate D_2. At the end of time T_1, item 1 satisfies its own demand and partial demand of item 2 for the duration T_2, till it is exhausted. Thus item 1 depletes at the rate $D_1 + (1-k_2)\ D_2$ during the time T_2 where b_2 units of item 2 is back-logged. At the end of T_2, item 1 and item 2 are jointly procured, and thus cycle repeats.

From Fig. 4.2, we establish following relations:

$$T_1 = \frac{Q_2 - b_2}{D_2} \tag{4.17}$$

$$T_2 = \frac{S_2}{D_1 + (1-k_2)\,D_2} \tag{4.18}$$

$$S_2 = Q_1 - D_1 ~_1 \tag{4.19}$$

$$b_2 = k_2 D_2 T_2 = \frac{k_2 D_2 S_2}{D_1 + (1+k_2) D_2} \tag{4.20}$$

Thus cycle time,

$$T = T_1 + T_2$$

$$= \frac{Q_1 + Q_2}{(D_1 + D_2)}$$

The various relevant cost per cycle are given as follows:

(g) Procurement Cost $= A + C_1 Q_1 + C_2 Q_2$

(h) Holding Cost:

For Item 1 $= \frac{1}{2} \frac{h_1}{(1-k_2)(D_1+D_2)^2} [\{(1-k_2)(D_1+D_2)$

$-k_2 D_2\} Q_1^2 - \frac{D_1}{D_2} (D_1 + \overline{1-k_2}\ D_2)\ Q_2^2$

$+2(D_1+D_2)(1-k_2)\ Q_1 Q_2]$ (4.21)

For Item 2 $= \frac{1}{2} \frac{h_2}{(1-k_2)(D_1+D_2)^2} [\{D_1 + D_2(1-k_2)\}^2 Q_2^2$

$+ k_2^2\ D_2^2\ Q_1^2 - 2k_2 D_1\ (D_1 + \overline{1-k_2}\ D_2) Q_1 Q_2)$

(4.22)

(i) Backlogging cost of Item 2:

$$= \frac{\pi_2 k_2\ (D_2 Q_1 - D_1 Q_2)}{(1-k_2)(D_1+D_2)} + \frac{1}{2} \frac{\tilde{\pi}_2}{(1-k_2)^2\ \bar{D}_2 (D_1+D_2)^2}$$

$$[D_2^{\ 2} Q_1^{\ 2} + D_1^{\ 2}\ Q_2^{\ 2} - 2D_1 D_2 Q_1 Q_2] \tag{4.23}$$

(j) Substitution Cost of Item 2 $= \frac{V_2}{(D_1+L_2)} (D_2Q_1 - D_1Q_2)$ (4.24)

Thus total annual cost is given by,

$$TC = \frac{1}{(Q_1+Q_2)} [A(D_1+D_2) + \{C_1(D_1+D_2) + V_2D_2\} Q_1 + \frac{\pi_2 k_2 D_2 Q_1}{(1-k_2)}$$

$$+ \{C_2(D_1+D_2) - V_2 D_1 - \frac{\pi_2 k_2 D_1}{(1-k_2)}\} Q_2 + \frac{Q_1^2}{2(1-k_2)(D_1+D_2)}$$

$$\{h_1((1-k_2)(D_1+D_2) - k_2D_2) + \frac{h_2 k_2^2 D_2}{(1-k_2)} + \frac{\tilde{\pi}_2 D_2}{(1-k_2)}\}$$

$$+ \frac{Q_2^2}{2(1-k_2)(D_1+D_2)} \{- \frac{h_1 D_1}{D_2} (D_1 + \overline{1-k_2}\, D_2)$$

$$+ h_2 \frac{(D_1 + \overline{1-k_2}\, D_2)^2}{D_2(1-k_2)} + \frac{\tilde{\pi}_2 D_1^2}{(1-k_2)D_2} \}$$

$$+ \frac{Q_1 Q_2}{2(1-k_2)(D_1+D_2)} \{h_1 (1-k_2)(D_1+D_2) - \frac{2h_2}{(1-k_2)}$$

$$x\ k_2(D_1 + \overline{1-k_2} D_2) - \frac{2\tilde{\pi}_2 D_1}{(1-k_2)} \}] \qquad (4.25)$$

The proportion of the backlogged demand of item 2 can be specified by management or it can be a decision variable.

<u>Case A:</u> k_2 is known.

Differentiating Eq. (4.25) with respect to Q_1 and setting it equal to zero, we get,

$$\frac{\partial TC}{\partial Q_1} = \frac{1}{(Q_1+Q_2)} [\ C_1(D_1+D_2) + \frac{\pi_2 k_2 D_2}{(1-k_2)} + \frac{Q_1}{(1-k_2)(D_1+D_2)}$$

$$\{h_1(1-k_2)(D_1+D_2) - D_1 k_2) + \frac{D 2k_2 h_2}{(1-k_2)} + \frac{\tilde{\pi}_2 D_2}{(1-k_2)} \} + V_2 D_2$$

$$+ \frac{Q_2}{2(1-k_2)(D_1+D_2)} \{(1-k_2)(D_1+D_2)\, h_1 -$$

$$- \frac{2k_2\, h_2}{(1-k_2)} (D_1+\overline{1-k_2}D_2) - \frac{2\tilde{\pi}_2\, D_1}{(1-k_2)}\}] - \frac{TC}{(Q_1+Q_2)^2} = 0$$

(4.26)

Differentiating Eq. (4.25) with respect to Q_2 and setting it equals to zero, we get,

$$\frac{\partial TC}{\partial Q_2} = \frac{1}{(Q_1+Q_2)} [C_2(D_1+D_2) - V_2D_1 - \frac{k_2D_1\pi_2}{(1-k_2)}$$

$$+ \frac{Q_2}{(1-k_2)(D_1+D_2)} \{- \frac{D_1}{D_2} (D_1 + \overline{1-k_2}\, D_2)\, h_1$$

$$+ \frac{(D_1 + \overline{1-k_2}\, D_2)^2}{(1-k_2)\, D_2} h_2 + \frac{D_1{}^2}{(1-k_2)D_2} \tilde{\pi}_2\}$$

$$+ \frac{Q_p}{2(1-k_2)(D_1+D_2)} \{(1-k_2)(D_1+D_2)\, h_1$$

$$- \frac{2k_2\, h_2}{(1-k_2)} (D_1+ \overline{1-k_2}\, D_2) - \frac{2\tilde{\pi}_2\, D_1}{(1-k_2)}\ ]$$

$$- \frac{TC}{(Q_1+Q_2)^2} = 0 \qquad (4.27)$$

Solving Eqs. (4.26) and (4.27), we get the optimal Q_1 in terms of Q_2

$$Q_1 = AQ_2 + B \qquad (4.28)$$

where,

$$A = C/F$$

$$B = E/F$$

$$C = -\{ \tfrac{3}{2} (1-k_2)\ D_1 + \tfrac{1}{2} (1-k_2)\ D_2 + \frac{D_1^{\ 2}}{D_2} \}\ h_1$$

$$+ \frac{D_1+(1-k_2)D_2}{(1-k_2)\ D_2} (D_1+D_2)\ h_2 + \frac{\tilde{\pi}_2\ D_1}{(1-k_2)\ D_2} (D_1+D_2)$$

$$E = -(C_1-C_2+V_2+\pi_2)\ (D_1+D_2)^2$$

$$F = h_1 \{ \tfrac{1}{2} (1-k_2)\ (D_1+D_2) - k_2D_2\}$$

$$+ \frac{k_2}{(1-k_2)} h_2(D_1+D_2) + \tilde{\pi}_2 \frac{(D_1+D_2)}{(1-k_2)}$$

By solving Eq. (4.26) and Eq. (4.28) we get the optimal value of Q_2. The positive values of Q_1 and Q_2 are put in Eq. (4.25) to get total annual cost TC. The total annual cost (TCIB) for joint replenishment of two items operating independently with back-logging is given by,

$$TCIB = C_1D_1 + C_2D_2 + \frac{h_1D_1T}{2} + \frac{h_2D_2T}{2} + \frac{(\tilde{\pi}_2+h_2)}{2D_2} \frac{b_2^{\ 2}}{T}$$

$$+ \frac{(\pi_2\ b_2 + A)}{T} - h_2\ b_2 \qquad (4.29)$$

where,

$$b_2 = (h_2 - \frac{\pi_2}{T}) \frac{D_2T}{(\tilde{\pi}_2+h_2)},$$

$$T = \left[\frac{2\{ \frac{(\tilde{\pi}_2+h_2)\ b_2^{\ 2}}{2D_2} + \pi_2b_2+A\}}{(h_1D_1 + h_2D_2)} \right]^{1/2}$$

It is compared with the total annual cost given by Eq. (4.25) and if,

$TC < TCIB$

The decision is taken to operate the coordinated replenishment system with partial back-logging and partial substitution of item 2 with item 1 and action taken is the procurement of the optimal quantities of item 1 and of item 2.

Case B: When k_2 is a decision variable.

Differentiate Eq. (4.25) with respect to k_2 and setting it equal to zero, we get,

$$\frac{\partial TC}{\partial k_2} = \frac{1}{(Q_1+Q_2)} [\{ \frac{\pi_2}{(1-k_2)} D_2 + \frac{k_2}{(1-k_2)^2} \pi_2 D_2 \} Q_1$$

$$- \frac{\pi_2 D_1 Q_2}{(1-k_2)^2} + \frac{Q_1^{\,2}}{2(D_1+D_2)} [- \frac{D_2 h_1}{(1-k_2)^2} + \frac{2k_2}{(1-k_2)^3} D_2 h_2$$

$$+ \frac{\tilde{\pi}_2 D_2}{(1-k_2)^3} + \frac{Q_2^{\,2}}{2(D_1+D_2)} [- \frac{1}{(1-k_2)^2} \frac{h_1 D_1^{\,2}}{D_2}$$

$$+ \frac{1}{(1-k_2)^3} \frac{D_1^2}{D_2} h_2 + \frac{1}{(1-k_2)^2} D_1 h_2 + \frac{1}{(1-k_2)^3} \frac{\tilde{\pi}_2 D_1^{\,2}}{D_2}]$$

$$+ \frac{Q_1 Q_2}{2(D_1+D_2)} [-2 \frac{(1+k_2)}{(1-k_2)^3} D_1 h_2 - \frac{2}{(1-k_2)^2} D_2 h_2$$

$$- \frac{2D_1 \tilde{\pi}_2}{(1-k_2)^3}] = 0 \qquad (4.30)$$

Provided $Q_1 + Q_2 > 0$ and $1-k_2 \neq 0$ we get k_2 in terms of Q_1 and Q_2

$$
\begin{aligned}
k_2 = \Big[& \pi_2(D_1Q_2 - D_2Q_1) + \frac{h_1}{2D_2(D_1+D_2)}\,(D_2^2\,Q_1^2 + D_1^2\,Q_1^2) \\
& + \frac{h_2}{2D_2(D_1+D_2)}\,(-D_1^2\,Q_2^2 - D_1D_2\,Q_1^2 + 2D_2(D_1+D_2))\,Q_1Q_2 \\
& - \frac{\tilde{\pi}_2}{2D_2(D_1+D_2)}\,(D_1Q_2 - D_2Q_1)^2\Big] \Big/ \Big[\pi_2(D_1Q_2 - D_2Q_1) \\
& + \frac{h_1}{2D_2(D_1+D_2)}\,(D_1Q_2 - D_2Q_1)^2 + \frac{h_2}{2(D_1+D_2)} \\
& \times (2D_2Q_1^2 - Q_2^2\,D_1 + 2D_2\,Q_1Q_2)\Big] \qquad (4.31)
\end{aligned}
$$

Solving Eqs. (4.26), (4.28) and (4.31), we get optimal values of Q_1, Q_2 and k_2. For positive values of Q_1 and Q_2 and $0 < k_2 < 1$, we calculate the total annual cost from Eq. (4.25). If TC < TCIB, we follow the decision rules obtained above.

$k_2 = 0$ corresponds to the case of complete substitution discussed in Sec. 4.2 while $k_2 = 1$ corresponds to the situation of complete back-logging of the demand of item 2.

CHAPTER V

JOINT REPLENISHMENT WITH CONTINUOUS SUBSTITUTION

5.1 Problem Statement:

In this chapter we consider the inventory systems sometimes prevailing in case of retailor, wholesaler/distributors in which we focus attention to stock two items of similar characteristics. If one of the items is newly introduced in the market, the retailer/whole-saler tries to substitute continuously the certain proportion of demand for the old product by the newly introduced product to catch market for the new item. Under such condition he attracts the customer by incurring extra cost in giving special sale coupons, prizes and advertising the product. Therefore, for such situation we consider the model where both items are jointly procured.

Our objective is to decide stocking policy of items for the system described above so that total annual operating cost is a minimum.

We shall develop the model for following two conditions:

1. No shortages
2. The shortages of old item are partially backlogged.

For the development of the model we retain the assumptions of Sec. 2.2 about demand rate, replenishment, planning horizon and unit variable costs. As mentioned in previous page, even if in stock situation, the demand of an item can be partially satisfied by other item (newly introduced), specifically we take item 2 as the item to be substituted by the other.

In addition to the notations used in Sec. (4.2.1), the following notation is introduced.

k = The proportion of the demand of item 2 satisfied by item 1, i.e. $0 < k < 1$.

5.2 Formulation:

This section deals with the formulation of the problem corresponding to both the cases specified in the previous section. The first case is dealt with in section 5.2.1 where as the second one is considered in Sec. 5.2.2.

5.2.1 No Shortages:

The inventory-level of item 1 and item 2 are shown in Fig. 5.1. Referring to Fig. we see that at the beginning of the cycle, Q_1 units of item 1 and Q_2 units of item 2 are jointly procured. Q_1 units of item 1 deplete with the rate $\{D_1 + (1-k)\ D_2\}$ in time T and Q_2 units of item 2 at the rate $k\ D_2$ in the time T. Thus, item 1 substitutes the demand of

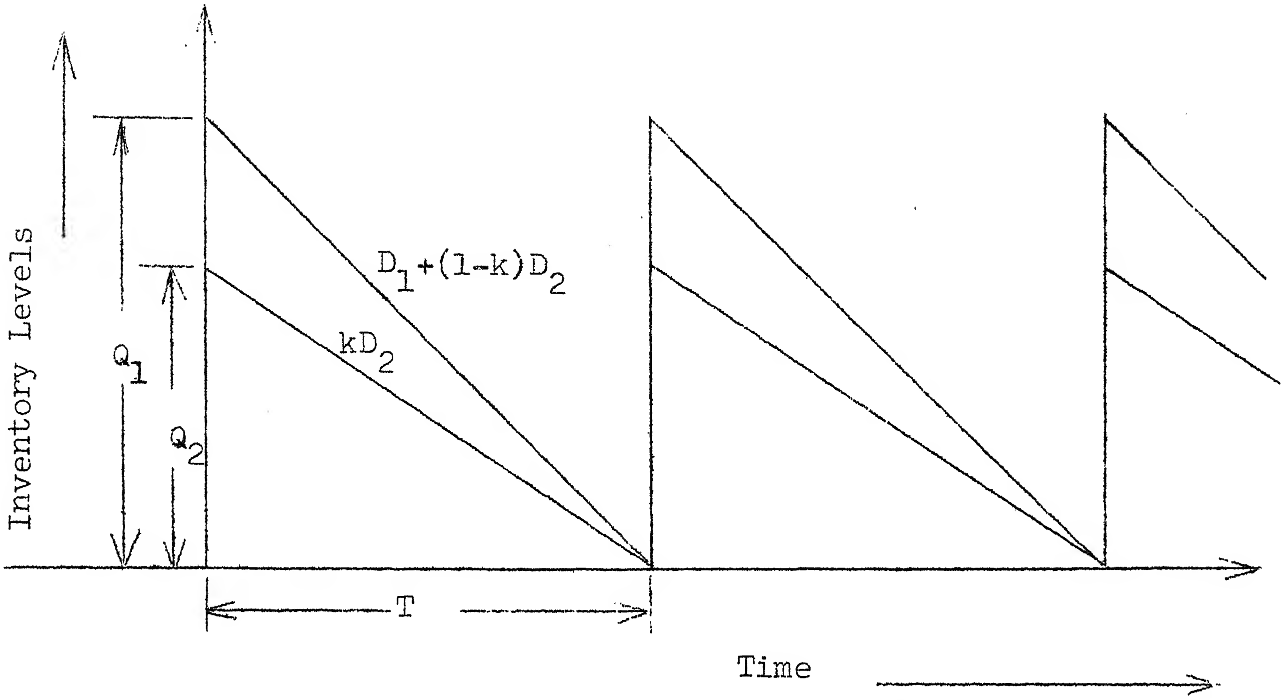

Fig. 5.1: Continuous substitution with no shortages.

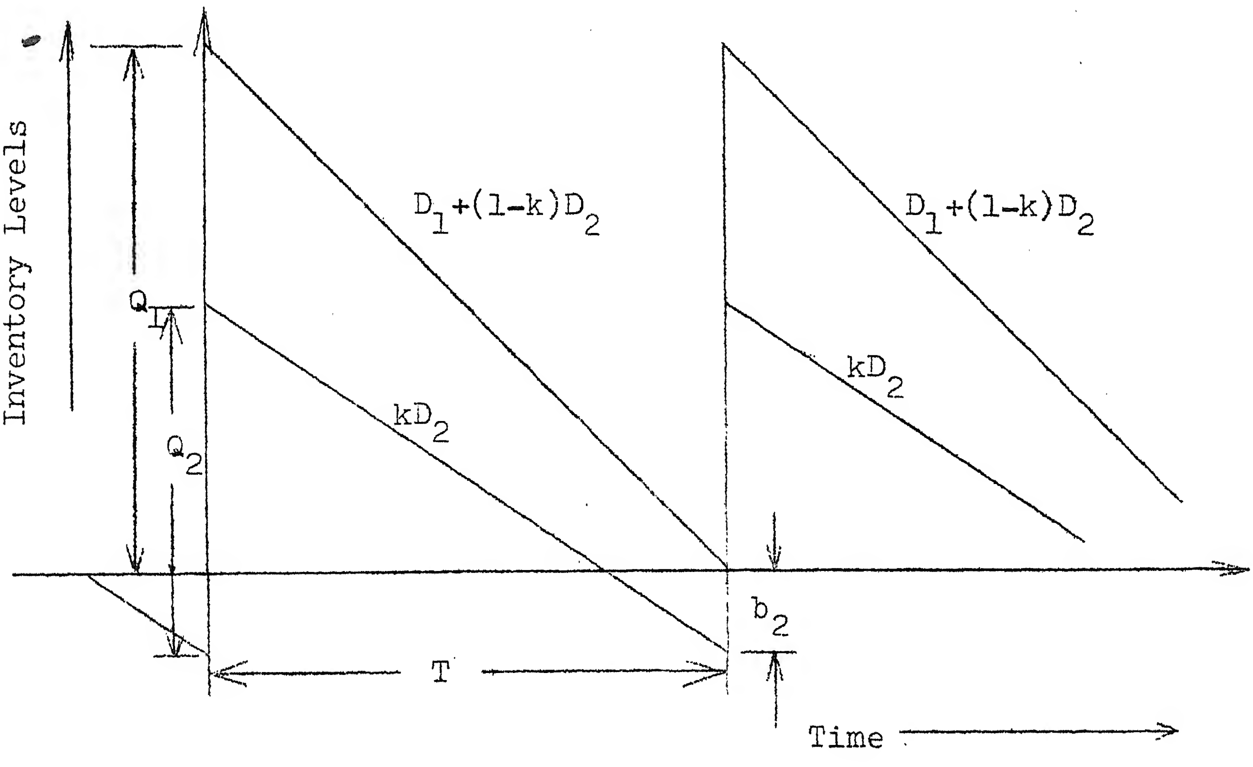

Fig. 5.2: Continuous substitution with partial backlogging.

item 2 with the rate $(1-k)D_2$ incurring additional substitution cost. At the end of time T again, item 1 and item 2 are jointly procured and thus, cycle repeats.

From the figure the following relations are easily derived.

$$T = \frac{Q_2}{kD_2} = \frac{Q_1}{D_1+(1-k)D_2} \tag{5.1}$$

Thus, cycle time

$$T = \frac{(Q_1+Q_2)}{(D_1+D_2)} \tag{5.2}$$

$$k = \frac{Q_2}{D_2 T} = [Q_2(D_1+D_2)]/D_2(Q_1+Q_2) \tag{5.3}$$

Various relevant cost per cycle are given as follows:

1. Procurement cost $= A + C_1Q_1 + C_2\ Q_2$ (5.4)

2. Holding cost of item 1 $= \frac{1}{2} h_1\ Q_1T$

$$= \frac{h_1Q_1(Q_1+Q_2)}{2(D_1+D_2)} \tag{5.5}$$

3. Holding cost of item 2 $= \frac{1}{2} h_2\ Q_2T$

$$= \frac{h_2Q_2(Q_1+Q_2)}{2(D_1+D_2)} \tag{5.6}$$

4. Cost of substitution $= (1-k)\ V_2D_1T$

$$= \frac{V_2(D_2Q_1 - D_1Q_2)}{(D_1+D_2)} \tag{5.7}$$

Total cost per cycle is sum of procurement cost, holding cost and cost of substitution. Therefore total annual cost is given by,

$$TC = \frac{1}{T}\left[A + C_1Q_1 + C_2Q_2 + \frac{h_1Q_1(Q_1+Q_2)}{2(D_1+D_2)} + \frac{h_2Q_2(Q_1+Q_2)}{2(D_1+D_2)} + \frac{V_2(D_2Q_1-D_1Q_2)}{(D_1+D_2)}\right]$$

$$= \frac{(D_1+D_2)}{(Q_1+Q_2)}A + \{C_1(D_1+D_2) + V_2D_2\}\frac{Q_1}{Q_1+Q_2} + \{C_2(D_1+D_2) - V_2D_1\}\frac{Q_2}{Q_1+Q_2} + \frac{1}{2}h_1Q_1 + \frac{1}{2}h_2Q_2 \tag{5.8}$$

For finding out the minimum total annual cost, differentiating Eq. (5.8) with respect to Q_1 and Q_2 respectively and setting them equal to zero, we get,

$$\frac{\partial TC}{\partial Q_1} = -\frac{(D_1+D_2)A}{(Q_1+Q_2)^2} - \frac{Q_1}{(Q_1+Q_2)^2}[C_1(D_1+D_2)+V_2D_2] + \frac{(C_1(D_1+D_2)+V_2D_2)}{(Q_1+Q_2)} - \frac{Q_2}{(Q_1+Q_2)^2} \times (C_2(D_1+D_2)-V_2D_1) + \frac{h_1}{2} = 0 \tag{5.9}$$

$$\frac{\partial TC}{\partial Q_2} = -\frac{(D_1+D_2)A}{(Q_1+Q_2)^2} - \frac{Q_1[C_1(D_1+D_2)+V_2D_2]}{(Q_1+Q_2)^2}$$

$$+ \frac{[C_2(D_1+D_2) - V_2D_1]}{(Q_1+Q_2)}$$

$$- \frac{Q_2}{(Q_i+Q_2)^2} [C_2(D_1+D_2)-V_2D_1] + \frac{1}{2} h_2 = 0 \tag{5.10}$$

Solving Eqs.(5.9) and (5.10), we get,

$$Q_1+Q_2 = \frac{2(C_1-C_2+V_2)(D_1+D_2)}{(h_2-h_1)} = kc \tag{5.11}$$

and hence cycle time

$$T = \frac{Q_1+Q_2}{D_1+D_2} = \frac{2(C_1-C_2 + V_2)}{(h_2-h_1)} \tag{5.12}$$

For $Q_1+Q_2 > 0$ we get the following conditions corresponding to three different cases,

i) if $h_2 > h_1$ then $C_1-C_2+V_2 > 0$

ii) if $h_2 < h_1$ then $C_1-C_2+V_2 < 0$ (5.13)

iii) if $h_2 = h_1$, the operating inventory system with interaction of the demand for two items does not give feasible solution and hence the model is not considered under this condition.

Solving Eq. (5.9) and Eq. (5.11), we get optimal Q_2 and Q_1 as follows:

$$Q_2 = \frac{2kc}{(h_2-h_1)} \left[\frac{A(D_1+D_2)}{kc^2} - \frac{h_1}{2} \right] \tag{5.14}$$

$$Q_1 = (kc - Q_2)$$
$$= kc\left[1 - \frac{2}{(h_2-h_1)}\left(\frac{A(D_1+D_2)}{kc^2} - \frac{h_1}{2}\right)\right] \tag{5.15}$$

5.2.2 Partial Backlogging of One item:

Along with continuous substitution of item 2, owing to customer preference to purchase item and specific relative values of cost parameters, in stock out situation, its shortages can be backlogged.

Following notations are introduced in addition to notations used in Sec. (4.2.1).

π_2 Unit backlogging cost of item 2 charged Rs/unit

$\tilde{\pi}_2$ Backlogging cost per unit per unit time of item 2

b_2 Backlogged units of item 2 in a cycle.

5.2.2.1 Model Development:

The inventory level of item 1 and item 2 are shown in Fig. 5.2. Referring to figure we see that at the beginning of the cycle Q_1 units of item 1 and Q_2 units of item 2 are jointly procured. (Q_2-b_2) units of item 2 are left on hand after satisfying b_2 backlogged units of item 2. Q_1 units of item 1 deplete with the rate $(D_1 + (1-k)D_2)$ in time (T_1+T_2) and (Q_2-b_2) units of item 2 deplete with rate

$k\,D_2$ in time T_1 and b_2 units of item 2 are backlogged in time interval of T_2. Thus the partial demand of item 2 is continuously satisfied by item 1 for time T. Again item 1 and item 2 are jointly procured and thus cycle repeats.

From the figure the following relations are easily established.

$$(Q_2 - b_2) = kD_2T_1 \tag{5.16}$$

$$b_2 = kD_2T_2 \tag{5.17}$$

$$Q_1 = (D_1+(1-k)D_2)\ T \tag{5.18}$$

Thus cycle time $$T = \frac{(Q_1+Q_2)}{(D_1+D_2)} \tag{5.19}$$

$$k = \frac{Q_2}{(Q_1+Q_2)}\ \frac{(D_1+D_2)}{D_2} \tag{5.20}$$

The various relevant costsper cycle are given as follows:

6. Procurement cost $= A + C_1Q_1+C_2Q_2$ (5.21)

7. Holding cost of item 1$= \frac{1}{2} h_1\ Q_1T = \frac{1}{2} h_1Q_1\ \frac{(Q_1+Q_2)}{(D_1+D_2)}$ (5.22)

8. Holding cost of item 2$= \frac{1}{2} h_2\ (Q_2-b_2)T_1$

$$= \frac{1}{2} h_2\ \frac{(Q_2-b_2)^2(Q_1+Q_2)}{Q_2(D_1+D_2)} \tag{5.23}$$

9. Backlogging cost of item 2 $= \pi_2 b_2$ (5.24)

10. Backlogging cost of item 2 $= \frac{1}{2}\tilde{\pi}_2 b_2 T_2 = \frac{\tilde{\pi}_2}{2}\frac{b_2^2}{Q_2}\frac{(D_1+D_2)}{(Q_1+Q_2)}$ (5.25)

11. Cost of substitution $= (1-k)\ D_2 V_2 T$

$$= V_2 \frac{(D_2Q_1-Q_2D_1)}{(D_1+D_2)} \tag{5.26}$$

Total cost per cycle is sum of procurement cost, holding cost, backlogging cost and the cost of substitution and therefore total annual cost is given by,

$$TC = \frac{(D_1+D_2)A}{(Q_1+Q_2)} + \frac{\{C_1(D_1+D_2)+V_2D_2\}Q_1}{(Q_1+Q_2)}$$
$$+ \frac{\{C_2(D_1+D_2) - V_2D_1\}\ Q_2}{(Q_1+Q_2)} + \frac{1}{2}\ Q_1h_1$$
$$+ \frac{1}{2}\ \frac{h_2(Q_2-b_2)^2}{Q_2} + \frac{\pi_2 b_2(D_1+D_2)}{(Q_1+Q_2)} + \frac{\tilde{\pi}_2\ b_2^{\ 2}}{2Q_2} \tag{5.27}$$

For finding out the minimum total annual cost, differentiating Eq. (5.27) with respect to Q_1, Q_2 and b_2 respectively and setting them equal to zero we get,

$$\frac{\partial TC}{\partial Q_1} = -\frac{(D_1+D_2)A}{(Q_1+Q_2)^2} - \frac{Q_1}{(Q_1+Q_2)^2}\{C_1(D_1+D_2)+V_2D_2\}$$
$$+ \frac{C_1(D_1+D_2)+V_2D_2}{(Q_1+Q_2)} - \frac{Q_2}{(Q_1+Q_2)^2}$$
$$\times \{C_2(D_1+D_2)-V_2D_1\} + \frac{1}{2}\ h_1 - \frac{\pi_2 b_2(D_1+D_2)}{(Q_1+Q_2)^2} = 0 \tag{5.28}$$

$$\frac{\partial TC}{\partial Q_2} = -\frac{(D_1+D_2)A}{(Q_1+Q_2)^2} - \frac{Q_1}{(Q_1+Q_2)^2}\left[C_1(D_1+D_2)+V_2D_2\right]$$

$$-\frac{Q_2}{(Q_1+Q_2)^2}\left[C_2(D_1+D_2)-V_2D_1\right]$$

$$+\frac{(C_2(D_1+D_2)-V_2D_1)}{(Q_1+Q_2)^2} + \frac{1}{2}h_2\left(1-\frac{b_2^2}{Q_2^2}\right)$$

$$-\frac{\pi_2 b_2(D_1+D_2)}{(Q_1+Q_2)^2} - \frac{\tilde{\pi}_2 b_2^2}{2Q_2^2} = 0 \qquad (5.29)$$

$$\frac{\partial TC}{\partial b_2} = -\frac{1}{2}h_2\frac{(Q_2-b_2)}{Q_2} + \frac{\pi_2(D_1+D_2)}{(Q_1+Q_2)} + \frac{\tilde{\pi}_2 b_2}{Q_2} = 0$$

or $$b_2 = \left[h_2 - \frac{\pi_2 D}{Q}\right]\frac{Q_2}{(\tilde{\pi}_2+h_2)} \qquad (5.30)$$

where, $D = D_1+D_2$

$Q = Q_1+Q_2$

From Eqs. (5.28) - (5.30), we get,

$$Q^2(h_2-h_1) - \left\{2(C_1-C_2+V_2) - \frac{\pi_2}{\tilde{\pi}_2+h_2}\right\}DQ - \frac{\pi_2^2 D^2}{\tilde{\pi}_2+h_2} = 0 \qquad (3.31)$$

$$Q = \Big[\left\{2(C_1-C_2+V_2) - \frac{\pi_2}{\tilde{\pi}_2+h_2}\right\}D$$

$$\pm\sqrt{\left[\left\{2(C_1-C_2+V_2) - \frac{\pi_2}{\tilde{\pi}_2+h_2}\right\}^2 D^2+4(h_2-h_1) \times \right.}$$

$$\left.\times \frac{\pi_2^2 D^2}{(\tilde{\pi}_2+h_2)}\right]\Big] / 2(h_2-h_1) \qquad (5.32)$$

Solving Eq. (5.28) and Eqs. (5.30)-(5.31) we get optimal value of Q_1, Q_2 and b_2.

Positive roots of Q_1, Q_2 and b_2 are taken for the feasible solution.

From Eq. (5.30), we get the following conditions for optimal value of $b_2 > 0$,

$$\frac{h_2}{\pi_2} > \frac{D}{Q} \tag{5.33}$$

CHAPTER VI

COORDINATED (MIXED) REPLENISHMENT WITH END SUBSTITUTION

6.1 Problem Statement:

Now we shall consider the inventory system where sometimes all items are procured and sometimes only few. This is widely known as Mixed or Coordinated Ordering Policy . We can consider the situation where in the stock-out condition the demand of an item can be substituted by other item. For our present purpose we shall be considering an inventory system consisting of two items, where better quality product can be used, if required, as a substitute of other. We thus, consider only one way substitution with coordinated replenishment of two items. We refer a better quality product as item 1 which is replenished all the time and available at every time, while other item is referred as item 2 which is not replenished all the time and is substituted by item 1 as and when necessary.

For the development of the model we retain the assumption of Sec. 2.2 of Chapter II about demand rate, replenishment, planning horizon, unit variable costs and shortages.

Furthermore, we assume that the amount of shortages of item 2 is less than for one period of the procurement of item 1 and shortages are in the last period of the procurement of item 1. And, the shortages of item 2 are completely satisfied by item 1. Furthermore, there are no shortages for item 1.

We introduce the following notations, in addition to notations used in Sec. (4.2.1).

A	Major set-up cost or procurement cost
a_i	Minor set-up cost of item i, for i = 1,2
S_2	On hand inventory of item 2 at n-th replenishment of item 1.
TRC	Total annual cost with coordinated replenishment policy with no substitution.
TC_{min}	Minimum total annual cost of coordinated replenishment with substitution.
$Q^{(1)}_{min}$	Units of item 1 to be procured at TC_{min}.
$Q^{(2)}_{min}$	Quantities of item 2 to be procured at TC_{min}.

6.2 Formulation:

The inventory levels of coordinated (mixed) replenishment with end substitution are shown in Fig. 6.1. At the beginning of the cycle, Q_1 units of item 1 and Q_2 units of item 2 are procured. Q_1 units of item 1 deplete at the rate D_1. As soon as, the inventory levels of item 1 goes to zero,

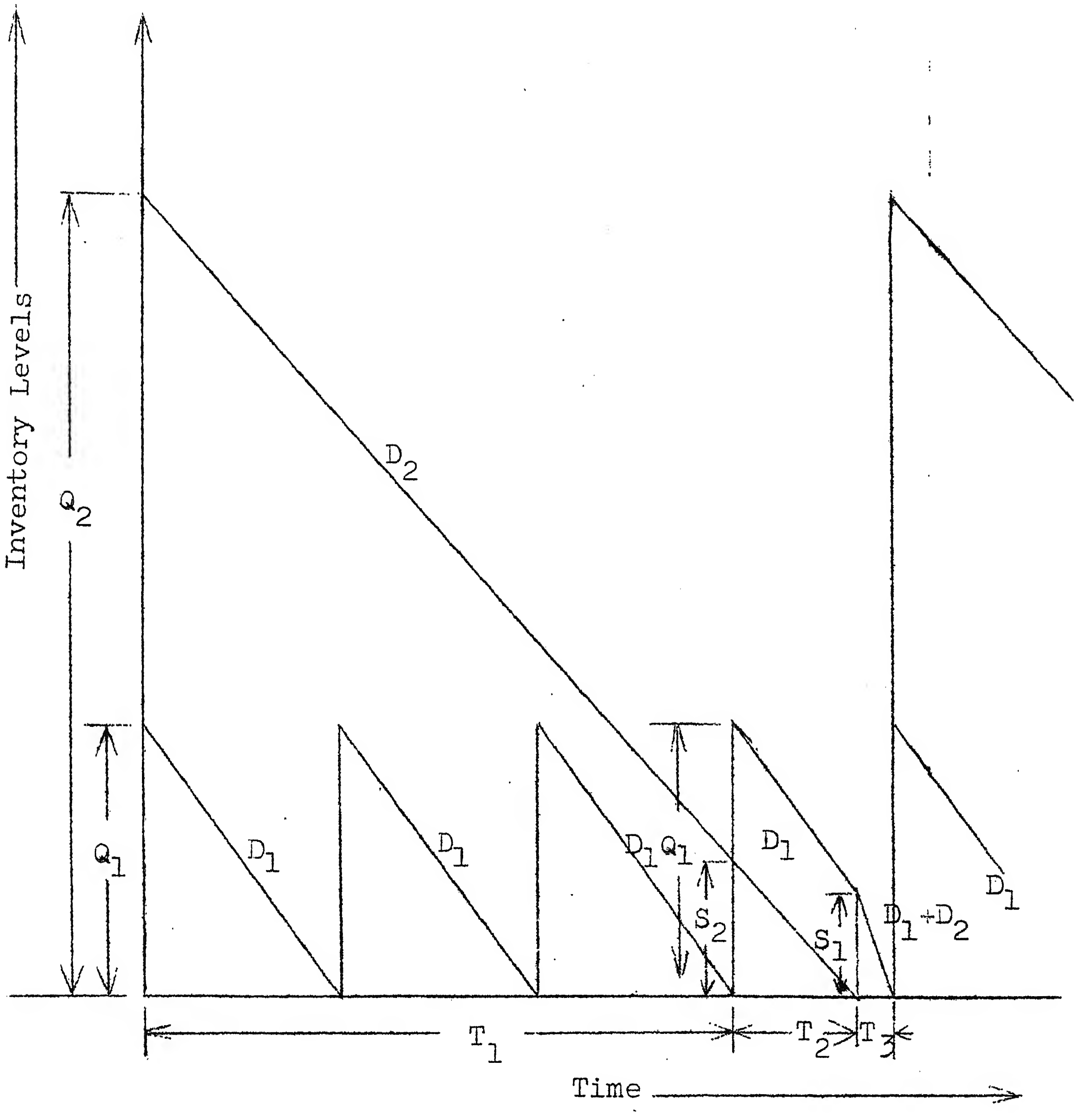

Fig. 6.1: Coordinated (mixed) replenishment with end substitution.

it is instantaneously replenished with Q_1 units. In time $T_1 + T_2$, item 1 is procured n times where $n \geq 1$. Q_2 units of item 2 deplete at the rate D_2 and item 2 is exhausted in time $T_1 + T_2$. The next procurement of both items is after time T_3. And thus item 2 is in shortage for the duration T_3. Therefore now, item 1 depletes at the rate $D_1 + D_2$ and thus satisfying its own demand as well as the demand of item 2 for duration T_3. At the end of T_3, item 1 and item 2 both are procured simultaneously. Thus, cycle repeats.

From the Fig. (6.1), we establish the following relations:

$$T_1 = (n-1)\frac{Q_1}{D_1} \tag{6.1}$$

$$T_2 = \frac{S_2}{D_2} = \frac{Q_2}{D_2} - (n-1)\frac{Q_1}{D_1} \tag{6.2}$$

$$T_3 = \frac{S_1}{(D_1+D_2)} = \frac{Q_1}{(D_1+D_2)} - \frac{D_1}{(D_1+D_2)}\left(\frac{Q_2}{D_2} - (n-1)\frac{Q_1}{D_1}\right) \tag{6.3}$$

Thus cycle time,

$$T = T_1 + T_2 + T_3 = \frac{Q_2 + nQ_1}{(D_1+D_2)} \tag{6.4}$$

The relevent costsper cycle are given as follows:

(A) Procurement Cost $= A + na_1 + a_2 + nC_1Q_1 + C_2Q_2$ (6.5)

(B) Holding Cost:

$$\text{For Item 1} = n \frac{[(D_1+D_2)-nD_2]}{2D_1(D_1+D_2)} Q_1{}^2 h_1 + n \frac{Q_1 Q_2 h_1}{(D_1+D_2)} - \frac{D_1 Q_2{}^2 h_1}{2D_2(D_1+D_2)} \tag{6.6}$$

$$\text{For Item 2} = \frac{Q_2{}^2}{2D_2} h_2 \tag{6.7}$$

$$\text{(C) Substitution Cost} = \frac{n\, V_2 D_2 Q_1 - V_2 D_1 Q_2}{(D_1+D_2)} \tag{6.8}$$

The total annual cost is given by,

$$\begin{aligned} TC(n) = \frac{1}{(Q_2+nQ_1)} [& \frac{(D_1+D_2)\ Q_2{}^2}{2D_2} (h_2 - \frac{D_1}{D_1+D_2} h_1) \\ & + \frac{n\ h_1\ (D_1-(n-1)\ D_2)}{2D_2} Q_1{}^2 + n\ Q_1 Q_2 h_1 \\ & + (A + na_1+a_2)\ (D_1+D_2) + nQ_1\ \{C_1(D_1+D_1)+V_2D_2\} \\ & + Q_2\ \{C_2(D_1+D_2) - V_2D_1\}] \end{aligned} \tag{6.9}$$

The decision variables are Q_1, Q_2 and n. Differentiating Eq. (6.9) with respect to Q_1 and setting it equal to zero, we get,

$$\frac{\partial TC(n)}{\partial Q_1} = \frac{1}{(Q_2+nQ_1)} \left[\frac{n(D_1-(n-1)D_2)}{D_1} Q_1h_1 + nQ_2h_1 \right.$$

$$\left. + n(C_1(D_1+D_2) + V_2D_2)\right] - \frac{nT(n)}{(Q_2+nQ_1)^2} = 0$$

(6.10)

Differentiating Eqs. (6.9) with respect to Q_2 and setting it equal to zero, we get,

$$\frac{\partial T(n)}{\partial Q_2} = \frac{1}{(Q_2+nQ_1)} \left[\frac{D_1+D_2}{D_2} Q_2 \left(h_2 - \frac{D_1}{D_1+D_2} h_1\right) + n\, Q_1h_1 \right.$$

$$\left. + C_2(D_1+D_2) - V_2D_1\right] - \frac{TC(n)}{(Q_2+nQ_1)^2} = 0 \quad (6.11)$$

From Eqs. (6.10) - (6.11), we got optimal $Q_2(n)$ in terms of $Q_1(n)$

$$Q_2(n) = -\frac{(n-1)\ Q_1h_1D_2}{D_1(h_2-h_1)} + \frac{(C_1-C_2+V_2)D_2}{(h_2-h_1)}$$

$$= AQ_1 + B \quad (6.12)$$

where,

$$A = -\frac{(n-1)\ h_1D_2}{D_1(h_2-h_1)}$$

$$B = \frac{(C_1-C_2+V_2)D_2}{(h_2-h_1)}$$

We get the following expression for optimal value of $Q_1(n)$,

$$E\, Q_1^2(n) + F\, Q_1(n) + G = 0 \tag{6.13}$$

where,

$$E = \frac{nh_1}{2} \left(1 - (n-1) \frac{D_2}{D_1}\right) + A^2 \left\{h_1 \frac{(D_1+2D_2)}{2D_2} - \frac{(D_1+D_2)}{2D_2} h_2 \right\} + h_1 \left\{1-(n-1) \frac{D_2}{D_1}\right\} A$$

$$F = 2AB \left\{h_1 \frac{(D_1+2D_2)}{2D_2} - \frac{(D_1+D_2)}{2D_2} \right) h_2 \Big\} + h_1 \left(1-(n-1) \frac{D_2}{D_1} \right\} + (C_1-C_2+V_2)\ (D_1+D_2)\ A$$

$$G = \left[h_1 \frac{(D_1 + 2D_2)}{2D_2} - \frac{(D_1+D_2)\ h_2}{2D_2} \right] B^2 + (C_1-C_2+V_2)(D_1+D_2)\ B - (A+na_1 + a_2)\ (D_1+D_2)$$

and thus solution of $Q_1(n)$ is given by,

$$Q_1^o(n) = - \frac{F \pm F^2-4EG}{2E} \tag{6.14}$$

The positive values of $Q_1(n)$ are taken and put in Eq. (6.12) to obtain the optimal value of $Q_2(n)$.

Solution Methodology:

The following steps are used to solve the problem.

Step 0: (Initialization) Set $n = 0$,

TC_{min} = Some high value

$Q^{(1)}_{min} = 0,$

$Q^{(2)}_{min} = 0.$

Step 1: Set $n = n+1$

Step 2: Solve for $Q_1(n)$ and $Q_2(n)$ from Eq. (6.14) and (6.12), respectively. Check $Q_1(n) > 0$, and $Q_2(n) > 0$, failing to which go to Step 5, otherwise,

$Q^{(1)}_{min} = Q_1(n)$

$Q^{(2)}_{min} = Q_2(n)$

Step 3: Calculate total annual cost, $TC(n)$, from Eq. (6.10).

Step 4: If condition $TC(n) > TC_{min}$ is satisfied, go to Step 5, otherwise, set $TC_{min} = TC(n)$, and go to Step 1.

Step 5: Set $n^* = (n - 1)$, stop.

It will give optimum value of $Q^{(1)}_{min}$, $Q^{(2)}_{min}$, n and TC_{min} for minimum cost, program.

Let the number of times procurement of item 1 n^* is made be denoted by m_1 and that of item 2 by m_2, note $m_1 = n^*$ and $m_2 = 1$. The total annual cost with coordinated replenishment

policy with no substitution is given by,

For item j, j = 1,2,

$$TRC\ (T^{*}, m_j' s) = \sum_{j=1}^{2} C_j m_j D_j + \frac{1}{2} \sum_{j=1}^{2}$$

$$+ \frac{A + \sum_{j=1}^{2} a_j/m_j}{T^{*}} \tag{6.15}$$

where,

$$T^{*} = \sqrt{\left[\frac{2(A + \sum_{j=1}^{2} a_j/m_j)}{\sum_{j=1}^{2} h_j\, m_j\, D_j} \right]}$$

Under coordinated replenishment, the substitution is profitable if,

$$TC_{min} < TRC\ (T, m_j\ s)$$

For the special case when n = 1, the procurement policy is the joint replenishment policy and the solution is same as discussed in Sec. (4.2).

CHAPTER VII

SUBSTITUTION IN CASE OF DYNAMIC DEMAND

7.1 Problem Statement:

So far, we have considered the inventory system where the nature of demand was static. Since, in most of the practical situation the demand for an item does not remain constant with respect to time, in this chapter we shall concentrate on the case in which the demand for the item is dynamic, but deterministic. This dynamic nature of the demand can be of two types. In first case, the demand can vary continuously with time and in second case the entire time span is divided into several equal intervals and the average demand is assumed to remain constant within each such interval.

We shall analyse the second case (discrete dynamic demand) for two-item inventory system with substitution of one item by the other, as and when required.

7.2 Assumptions:

The following assumptions are used to characterise the problem:

1. The demand rate in period j is denoted by D_j and planning horizon is divided into N periods.

2. There is no discount on the replenishment quantities.

3. The carrying cost is only applicable to inventory which is carried over from one period to the next.

4. The demand in an out-of-stock condition is backordered and can be resupplied upon the receipt of next procurement of the same item or the other item by evaluating the costs of backordering and substitution.

7.3 Notations:

We introduce the following notations:

For item i, $i = 1,2,.$

$A_t^{(i)}$ Fixed cost of procurement of item i for period t

$C_t^{(i)}$ Unit cost of item i for period t

$h_t^{(i)}$ Holding cost of item i for period t in Rs./unit.

$\pi_i^{(i)}$ Unit back-ordering cost of item i for period t, charged Rs./unit

$V_t^{(i)}$ Substitution cost of item i for period t, charged Rs./unit.

$D_t^{(i)}$ Demand of item i for period t

$Q_t^{(i)}$ The replenishment quantities of item i in period t.

i_i The period upto which item i is procured in the last replenishment.

j_i The period upto which the demand of item i is backlogged i.e. replenishment period for item i is $j_i + 1$.

k_i The period upto which item i is procured.

$I_{k_1k_2}$ The cumulative inventory position of both items, for periods 1,2,..., k_1 for item 1 and for periods 1,2,..., k_2 for item 2.

$M^0_{i_1i_2j_1j_2k_1k_2}$ The cost incurred for procuring item 1 in period j_1+1 to satisfy the demand from period i_1+1 to k_1 and item 2 in period j_2+1 to satisfy the demand from period i_2+1 to k_2 with no substitution.

$M^2_{i_1i_2j_1j_2k_1k_2}$ The cost incurred for procuring item 1 in period j_1+1 to satisfy its demand from period i_1+1 to k_1 and the demand of item 2 from i_2+1 to j_1+1 and procuring item 2 in period j_2+1 to satisfy its demand from period j_1+2 to k2.

The other intermediate notation will be introduced during formulation.

7.4 Formulation:

Let item 1 be procured in period j_1+1 to satisfy the demands from period i_1+1 to k_1 and item 2 is replenished in period j_2+1 for satisfying demands from period i_2+1 to k_2. Thus the demands of item 1 are backlogged from the period i_1+1 to j_1 and the demands of item 2 are backlogged from period i_2+1 to j_2.

When there is no substitution of the demand and both items are procured independently for periods i_1+1 to k_1 and

i_2+1 to k_2 respectively. The cost for these periods is the sum of procurement cost, holding cost and backlogging cost of both items. This cost is given by,

$$M^{o}_{i_1 i_2 j_1 j_2 k_1 k_2} = \sum_{i=1}^{2} [A^{(i)}_{j_i+1} + C^{(i)}_{j_i+1} \sum_{is_i=i_i+1}^{k_i} D^{(i)}_{is_i}$$
$$+ \sum_{js_i=j_i+1}^{k_i-1} h^{(i)}_{js_i} \sum_{iu_i=js_i+1}^{k_i} D^{(i)}_{iu_i}$$
$$+ \sum_{is_i=i_i+1}^{j_i} D_{is_i} \sum_{iu_i=is_i}^{j_i} \pi^{(i)}_{iu_i}] \quad (7.1)$$

and quantity of item i procured,

$$Q^{(i)}_{j_i+1} = \sum_{is_i=i_i+1}^{k_i} D^{(i)}_{is_i} \qquad \text{for } i = 1,2 \qquad (7.2)$$

and $M^{o}_{i_1 i_2 j_1 j_2 k_1 k_2} = 0, \quad Q^{(i)}_{j_1+1} = 0$

When item 1 is procured before item 2 in period j_1+1, it satisfies its demand from period i_1+1 to k_1 as well as the demand of item 2 from period i_2 to j_1+1 and item 2 is procured from period j_1+2 to k_2. Thus the demand of item 1 is backlogged from period i_1+1 to j_1 and the demand of item 2 is backlogged from period i_2+1 to j_1 and then j_1+2 to j_2. Therefore, total replenishment quantities of item 1 are the sum of the demands for item 1 from period i_1+1 to k_1 and the demands for item 2 from the period i_2+1 to j_1+1 and the replenishment

quantities of item 2 are the demands for item 2 from j_1+2 to k_2. Then the cost for these period which also includes substitution cost, is given by,

$$M^2_{i_1 i_2 j_1 j_2 k_1 k_2} = \sum_{i=1}^{2} [A^{(i)}_{j_i+1} + C^{(i)}_{j_i+1}\, Q^{(i)}_{j_i+1}]$$
$$+ \sum_{js_1=j_1+1}^{k_1-1} h^{(1)}_{js_1} \sum_{iu_1=js_1+1}^{k_1} D^{(1)}_{iu_1}$$
$$+ \sum_{is_1=i_1+1}^{j_1} D^{(1)}_{is_1} \sum_{iu_1=is_1}^{j_1} \pi^{(1)}_{iu_1}$$
$$+ \sum_{js_2=j_2+1}^{j_2} h^{(2)}_{js_2} \sum_{iu_2=js_2+1}^{k_1} D^{(2)}_{iu_2}$$
$$+ \sum_{is_2=i_2+1}^{j_1} D^{(2)}_{is_2} \sum_{iu_2=is_2}^{j_1} \pi^{(2)}_{iu_2}$$
$$+ \sum_{is_2=j_1+2}^{j_2} D^{(2)}_{is_2} \sum_{iu_2=is_2}^{j_2} \pi^{(2)}_{iu_2} \qquad (7.3)$$

and quantity of item 1 to be procured,

$$Q^{(1)}_{j_1+1} = \sum_{is_1=i_1+1}^{k_1} D^{(1)}_{is_1} + \sum_{is_2=i_2+1}^{j_1+1} D^{(2)}_{is_2} \qquad (7.4)$$

Similarly the quantity of item 2 procured is given by,

$$Q^{(2)}_{j_2+1} = \sum_{is_2=j_1+2}^{k_2} D^{(2)}_{is_2} \qquad (7.5)$$

7.5 Optimizing Algorithm:

For the solution of the problem we develop the following algorithm based on the dynamic programming. This algorithm is similar to H.M. Wagner and T.M. Whitin algorithm [7] which guarantees optimal replenishment quantities of single item with the assumptions of no shortages and zero ending inventory. Using the proposed algorithm, we can determine the lot sizes for each item i.e., $Q_1^{(1)}$, $Q_1^{(2)}$, $Q_2^{(1)}$, $Q_2^{(2)}$, ..., $Q_n^{(1)}$, $Q_n^{(2)}$ for the minimum value of total cost which is the sum of procurement cost, inventory carrying cost, backordering cost and substitution cost.

Since ending inventories of item 1 and item 2 are zero in periods i_1 and i_2 and in periods k_1 and k_2 respectively.

$$I_{i_1 i_2} = 0$$

$$I_{k_1 k_2} = 0$$

When there is no substitution, the on hand inventory for item 1 is positive from period j_1+1 to k_1-1 and negative from period i_1+1 to j_1. Similarly for item 2 the on-hand inventory is positive from period j_2+1 to k_1-1 and negative from i_2+1 to j_2. Therefore, the cost incurred and replenishment quantities from period i_1+1 to k_1 for item 1 and from i_2+1 to k_2 for item 1 and from i_2+1 to k_2 for item 2 are given by Eqns. (7.1) and (7.2).

When there is substitution of demand of item 2 from periods i_2+1 to j_1+1 by item 1, the cost, incurred and replenishment quantities of both the items are given by Eqns. (7.3) - (7.5).

Similarly for the substitution of demand for item 1 from i_1+1 to j_2+1, the costs incurred and the replenishment quantities of both the items are given by Eqns. (7.3) to (7.5) by inter-changing indices.

With this definition of $M^c_{i_1 i_2 j_1 j_2 k_1 k_2}$ we can write the following recursive relation for $F_{k_1 k_2}$.

$$F_{k_1 k_2} = \min_{\substack{0 \leq i_1 \leq j_1 < k_1 \\ 0 \leq i_2 \leq j_2 < k_2}} \left[F_{i_1 i_2} + M^c_{i_1 i_2 j_1 j_2 k_1 k_2} \right] \quad (7.6)$$

$$\text{for } k_1 = 1, 2, \ldots, n, \qquad k_2 = 1, 2, \ldots, n$$

where $F_{00} = 0$, $c = 0, 1, 2$, indicating no substitution, substitution of 1 by 2 and substitution of demand for item 2 by item 1 respectively.

Eq. (7.6) can be explained as follows:

For k_1 period horizon for item 1 and k_2-period for item 2 with zero initial and final inventories there will be some periods i_1 and i_2 upto which last procurement of item 1 and item 2 is made. Assuming that we have found the optimal policy and hence minimum cost for $F_{i_1 i_2}$ for every $i_1 < k_1$

and $i_2 < k_2$ where $I_{i_1 i_2} = 0$, we can compute $M_{i_1 i_2 j_1 j_2 k_1 k_2}$. The minimum cost for k_1, k_2 periods horizon results from selecting the optimal period upto which last procurement of item 1 and item 2 are made. By trying all $i_1 < k_1$, $i_2 < k_2$, $j_1 < k_1$ and $j_2 < k_2$ we can find the value of (i_1, i_2), (j_1, j_2), say (i_1^*, i_2^*), (j_i^*, j_2^*) which minimizes Eq. (7.5). Thus the replenishment periods are j_1+1 and j_2+1 for item 1 and item 2, respectively. The procedure is adapted to find the $F_{k_1 k_2}$ for $k_1 = 1, \ldots, n$, $k_2 = 1, \ldots, n$ when $F_{n,n}$ is found. We have minimum cost value for the n-period horizon and can use (i_1, i_2) to work back to extract optimal lot sizes.

To be precise we can write the algorithm as follows:

<u>Step 1</u>: Vary k_1 from 1 to n in steps of 1,
and k_2 from 1 to n in steps of 1,
vary i_1 from 0 to k_1-1
and i_2 from 0 to k_2-1 in steps of 1,
vary j_1 from i_1 to k_1-1
and j_2 from i_2 to k_2-1 in steps of 1.

<u>Step 2</u>: If $j_1 = j_2$, $k_1 < k_2$ and $j_1+1 = k_1$, substitute the demand of item 1 by item 2 and calculate $M^c_{i_1 i_2 j_1 j_2 k_1 k_2}$, $Q^{(1)}_{j_1+1}$ and $Q^{(2)}_{j_2+1}$ from Eqns. (7.3) - (7.5) by interchanging the indices. Give indices 1 to a counter to indicate that the demand of item 1 is substituted

by item 2, otherwise,

If $k_1 = k_2$, $j_1+1 = k_1$, and $j_1 = j_2$.
Calculate $M^2_{i_1 i_2 j_1 j_2 k_1 k_2}$, $Q^{(1)}_{j_1+1}$ and $Q^{(2)}_{j_2+1}$ from Eqs. (7.3) - (7.5) with counter = 2, indicating the demand of item 2 is substituted by item 1; calculate $M^1_{i_1 i_2 j_1 j_2 k_1 k_2}$, $Q^{(1)}_{j_i+1}$ and $Q^{(2)}_{j_2+1}$ using Eqs. (7.3) - (7.5) with counter = 1, otherwise,

If $k_1 > k_2$, $j_2+1 = k_2$ and $j_1 = j_2$,
calculate $M^1_{i_1 i_2 j_1 j_2 k_1 k_2}$ from Eqs. (7.3) - (7.4) with counter = 1.

Step 3: Calculate the $M^0_{i_1 i_2 j_1 j_2 k_1 k_2}$, $Q^{(1)}_{j_1+2}$ and $Q^{(2)}_{j_2+1}$ from Eqs. (7.1) and (7.2) with counter = 0.

Step 4: Calculate $F_{k_1 k_2}$ from Eq. (7.6).

Step 5: Continue until $F_{n,n}$ is evaluated.

Step 6: Track-back optimal i_1, i_2, j_1, j_2, $k_1 k_2$ for minimum total cost for n periods and final decision for quantities to be procured substituted amount and back-logged demand of both items in different periods.

Numerical Example:

Consider a 8 periods problem for two items. The relevant data for the problem are given in Table 7.1

Solution: Using the various steps of the algorithm, we obtain final values of decision variables and the relevant costs as given in Table 7.2(a) and 7.2(b).

Table 7.1:Basic Data for Numerical Example

PERIODS	DEMANDS ITEM1	ITEM2	UNIT COST ITEM1	ITEM2	SUBSTITUTION COST ITEM1	ITEM2
1	500.0	450.0	25.0	23.0	0.5	0.4
2	550.0	500.0	23.0	24.0	0.5	0.5
3	500.0	700.0	24.0	22.0	0.6	0.8
4	700.0	600.0	23.5	21.0	0.9	0.9
5	550.0	500.0	24.0	24.0	0.8	0.6
6	550.0	700.0	25.0	25.0	0.5	0.6
7	800.0	900.0	23.0	22.5	0.9	0.7
8	600.0	700.0	24.0	23.0	0.4	0.4

Table 7.1 (continued)

PERIODS	FIXED COST OF ITEM1	ITEM2	HOLDING COST ITEM1	ITEM2	BACKLOGGING COST ITEM1	ITEM2
1	200.0	250.0	6.0	5.5	4.5	5.5
2	225.0	200.0	5.5	6.0	6.0	5.5
3	200.0	200.0	5.0	5.0	5.5	5.0
4	250.0	225.0	5.0	5.5	5.5	5.0
5	200.0	250.0	6.0	6.0	5.0	5.5
6	225.0	225.0	5.5	6.0	6.0	5.5
7	250.0	300.0	5.8	4.0	5.0	4.5
8	350.0	400.0	6.5	5.5	5.5	5.0

Table 7.2(a):Optimal Values of

I1	I2	J1	J2	K1	K2	COUNTER	TOTAL COST F(K1,K2)
0	0	0	0	1	1	1	22350.0
1	1	1	1	2	2	2	49275.0
2	2	2	2	3	3	1	76175.0
3	3	3	3	4	4	1	104330.0
4	4	4	4	5	4	0	120130.0
5	4	5	4	5	5	0	132380.0
5	5	5	5	5	6	2	154275.0
6	6	6	6	7	5	0	182925.0
7	6	7	6	7	7	0	203475.0
7	7	7	7	8	8	1	234015.0

Note: Counter zero indicates the case of no substitution.
Counter one indicates the case of subtituion of item 1 by item 2.
Counter two indicates the case of substitution item 2 by item 1.

Table 7.2(b):Optimal Solution of the Numerical Example

PERIODS	QUANTITY PROCURED OF		SUBSTITUTED QUANTITY OF		BACKLOGGED UNITS OF	
	ITEM1	ITEM2	ITEM1	ITEM2	ITEM1	ITEM2
1	0.0	950.0	500.0	0.0	0.0	0.0
2	1150.0	0.0	0.0	500.0	0.0	0.0
3	0.0	1200.0	500.0	0.0	0.0	0.0
4	0.0	1300.0	700.0	0.0	0.0	0.0
5	650.0	500.0	0.0	0.0	0.0	0.0
6	1250.0	0.0	0.0	700.0	0.0	0.0
7	800.0	900.0	0.0	0.0	0.0	0.0
8	0.0	1300.0	600.0	0.0	0.0	0.0

7.6 Discussions:

The results show that sometimes it is preferred to procure item 1 to substitute the demand for item 2 and sometimes, item 2 to satisfy its own demand as well as the demand of item 1.

From the Table 7.3 we see that the CPU increases exponentially with the number of stage n on solving the examples on DEC - 1090 computer system.

Table 7.3: CPU time for n-stage problems

n	CPU
3	.17
4	.24
5	.38
6	.67
7	1.24
8	2.13

In addition to what we have done, it is possible to find that period upto which it is economical to substitute item 2 by item 1, even beyond the replenishment period of item 1 and before the replenishment period of item 2.

Let j_2' be the period upto which item 1 is procured to satisfy its demand as well as the demand of item 2 such that

$$Q^{(2)}_{j_2+1} = \sum_{is_2=j_2'+1}^{k_2} D^{(2)}_{is_2} \tag{7.9}$$

If $j_2 \leq j_1$, and item 1 is substituted by item 2, the similar expression for the cost and procurement quantities of both items can be given by Eqs. (7.7) - (7.9) interchanging the indices. To incorporate the condition we can suitably modify the algorithm at Step 2 for testing $j_1 < j_2$ and $j_2 < j_1$. For given i_1, i_2, j_1, j_2, k_1 and k_2 such that $j_1 < j_2$ varying j_2' in Step of 1 such that $j_1+1 \leq j_2 < j_2$. We can find minimum $M^2_{i_1i_2j_1j_2k_1k_2}$, $Q^{(1)}_{j_1+1}$ and $Q^{(2)}_{j_2+1}$ using Eqs. (7.7) - (7.9) and other steps remain same.

CHAPTER VIII

n - ITEMS SUBSTITUTION

In this chapter we study an inventory system consisting of n items with similar characteristics in any of these can be used to satisfy the demand of any other item. The objective now is to select the items to satisfy the demands of all n items with minimum total costs. In general, the number of such selected items would be between 1 to n.

8.1 Assumptions and Notations:

For the development of model we retain the assumptions of section 2.2. about demands, replenishment, planning horizon and variable costs. The not tions used are given as follows:

n Total number of items

For item $i = 1, \ldots, n$

D_i Annual demand rate of i-th item

A_i Fixed ordering cost of i-th item

C_i Unit cost of item i

h_i Holding cost per unit per unit time of the item

V_{ij} Cost of per unit substitution of item i by item j

v_i Unit substitution cost of i-th item when it is substituted by any other item.

Q_i Order quantity for i-th item.

Other intermediate notations are used during formulation.

8.2 Formulation:

First we shall consider the case of two items with complete substitution of one item by the other. The inventory level for such case is shown in Fig. 8.1. Let item 1 be selected for the purpose to substitute the demand of item 2.

Q_1 units of item 1 procured at the beginning of cycle deplete at the rate $(D_1 + D_2)$ in the cycle period T indicating that demand of item 2 is completely substituted by item 1. Then,

$$\text{Cycle time } T = \frac{Q_1}{(D_1+D_2)} \tag{8.1}$$

and the total annual cost is given, by

$$TC_1 = \frac{(D_1-D_2)}{Q_1}\left[A_1 + C_1Q_1 + \frac{V_2 D_2}{(D_1+D_2)} Q_1 + \frac{1}{2} h_1 Q_1^{\,2}\right]$$

$$= \frac{A_1(D_1+D_2)}{Q_1} + C_1\,(D_1+D_2) + V_2 D_2 + \frac{1}{2} h_1 Q_1 \tag{8.2}$$

The decision variable Q_1 for minimum value of total annual cost with suhstitution is given by,

$$Q_1^* = \left[\frac{2A_1\,(D_1+D_2)}{h_1}\right]^{1/2} \tag{8.3}$$

Thus we see that for complete substitution of the demand of one item by the other, the optimal procurement quantity of the

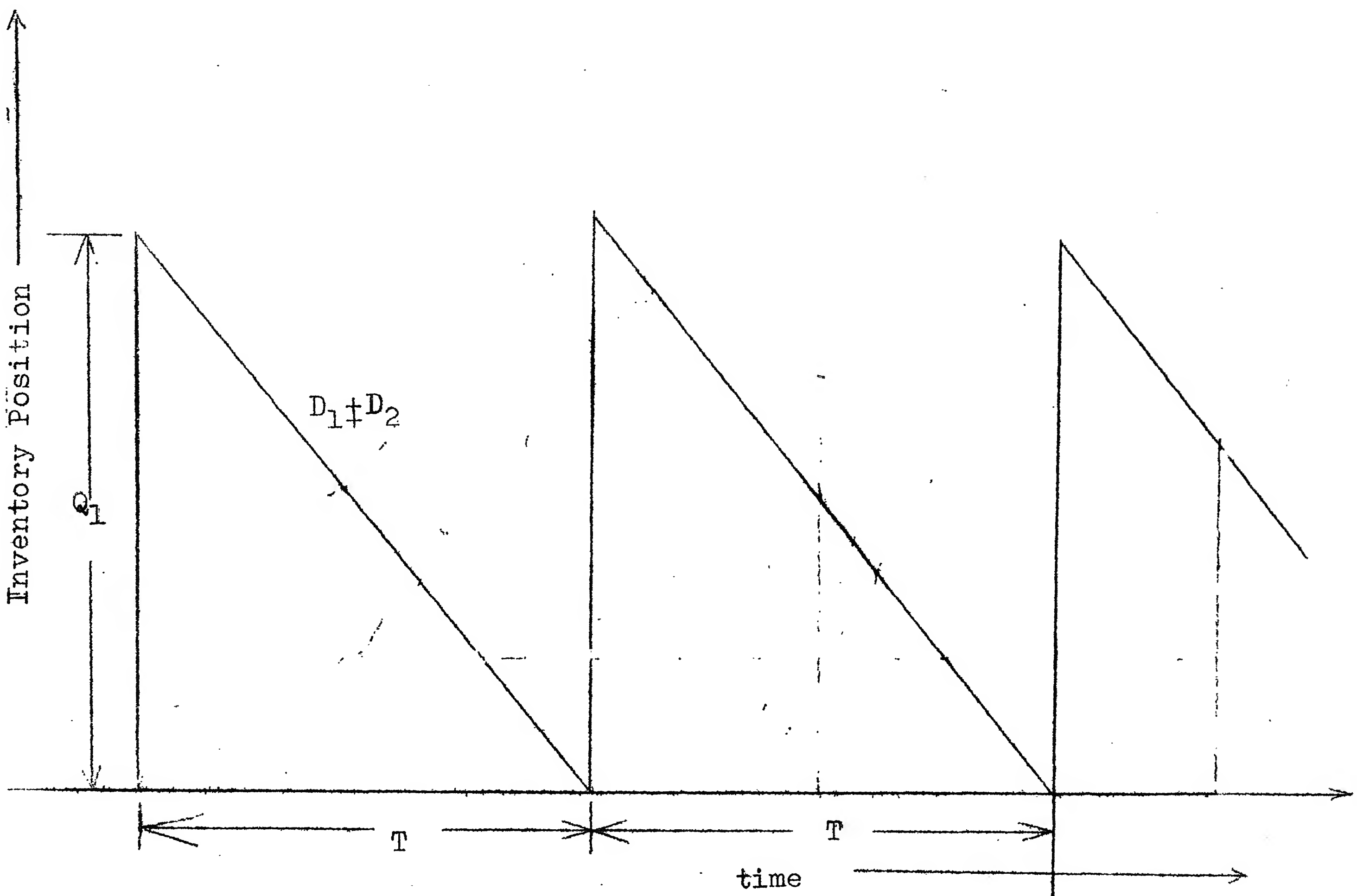

Fig :8.1 :Complete Substitution of Demand of One Item by the Other.

item is given by EOQ with annual demand rate equal to the sum of demands of both the items. And minimum total annual cost is given by,

$$TC_1^* = C_1(D_1+D_2) + V_2D_2 + \{2A_1h_1(D_1+D_2)\}^{1/2} \qquad (8.4)$$

which is the sum of total annual relevant costs of an item obtained under EOQ model, annual procurement and substitution cost.

When two items are operating independently to satisfy their demands (that is, no substitution), the annual cost is given by,

$$TC_0^* = \sum_{i=1}^{2} \{C_1 D_i + (2A^i h^i D^i)^{1/2}\} \qquad (8.5)$$

Similarly, when item 2 satisfies the demand of item 1 completely, in addition to its own demand, the total annual cost is given by,

$$TC_2^* = C_2(D_1+D_2) + V_1D_1 + (2A_2h_2(D_1+D_2))^{1/2} \qquad (8.6)$$

The item 1 would substitute the demand of item . 2 iff,

$$TC_1^* \leq \min (TC_0^* , TC_2^*) \qquad (8.7)$$

Now we shall consider the following case.

Case I: All cost parameters and demands are identical for two items, i.e.,

$$h_1 = h_2 = h , \quad C_1 = C_2 = C, \quad V_1 = V_2 = V,$$
$$D_1 = D_2 = D , \quad A_1 = A_2 = A.$$

The total cost obtained from Eqs. (8.4) and (8.6) are same. Therefore, to substitute one item by other we get the following condition,

$$2CD + VD + \sqrt{4ADh} \leq 2CD + 2\sqrt{2ADh}$$

or
$$VD \leq (2 - \sqrt{2})\sqrt{2ADh} \tag{8.8}$$

Eq. (8.8) gives the condition to substitute completely the demand of one item by the other.

Now we generalize the above treatment to the case of n-items.

8.3 Case of n-Identical Item:

Let us consider three items first. Suppose we have taken decision to substitute the demand of item 2 with item 1, and now we consider to substitute the demand of third item by item 1. Item 3 is substituted by item 1 iff,

$$3CD + 2VD + \sqrt{6ADh} \leq 3CD + VD + \sqrt{4ADh} + \sqrt{2ADh}$$

$$VD \leq \{(\sqrt{2} + 1) - \sqrt{3}\}\sqrt{2ADh} \tag{8.9}$$

Similarly, for the substitution of the demand of m-th item by item 1 when we have taken decision to substitute (m-1)-th item by item 1, we get the condition,

$$m\ CD + (m-1)\ VD + \sqrt{2m\ ADh} \leq m\ CD + (m-2)\ VD + \sqrt{2(m-1)\ ADh} + \sqrt{2ADh}$$

or
$$VD \leq (1 + \sqrt{m-1} - \sqrt{m})\sqrt{2ADh}$$

$$\text{for } m = 2, \ldots, n \tag{8.10}$$

8.4 Case of Different Substitution Costs:

We now consider a case where unit substitution cost for each item is different and other parameters are same for all items. Without loss of generality the indices are given to items in increasing order of the unit substitution cost,

$$V_1 \leq V_2 \leq V_3 \leq \dots \leq V_n$$

The following one of the two conditions which are general form of Eqs. (8.9) and (8.10) must, be satisfied in order to satisfy the demand of m-th item by item 1, when we have decided to substitute the demands for item 2 to (m-1) by item 1.

Substitution cost for m-th item substitution,

$$V_m D \leq (1 + \sqrt{(m-1)} - \sqrt{m}) \sqrt{2ADh} \quad \text{for } m = 2, \dots, n \qquad (8.11)$$

and total substitution cost, for (m-1) items substitution by item 1

$$\left(\sum_{i=2}^{m} V_i \right) D \leq (m - \sqrt{m}) \sqrt{2ADh} \qquad m = 2, \dots, n \qquad (8.12)$$

If Eq. (8.11) or (8.12) is not satisfied for m-th item the decision is taken not to substitute its demand by item 1. In other words the unit substitution cost of m-th item should be less than or equal to $\left(\frac{1}{D} (1 + \sqrt{m-1} - \sqrt{m}) \sqrt{2ADh} \right)$.

8.5 Different Purchase Cost:

The unit cost of item differs for each item

and all other costs, including unit substitution costs and demands are same. The item indices are given in increasing order of unit cost of item.

For this case, Eq. (8.10) is rewritten as:

For m-th Item Substitution:

$$mC_1D + (m-1)\ VD + \sqrt{2mADh} \le (m-1)\ C_1D + C_mD + (m-2)VD + \sqrt{2(m-1)ADh} + \sqrt{2ADh}$$

$$VD \le (C_m - C_1)\ D + (1 + \sqrt{m-1} - \sqrt{m})\sqrt{2ADh} \qquad \text{for } m = 2, \ldots n \qquad (8.13)$$

Since $C_m > C_{m-1}$, therefore,

$$(C_m - C_1)\ D + (1 + \sqrt{(m-1)} - \sqrt{m})\ \sqrt{2ADh} > (C_2 - C_1)\ D + (2 - \sqrt{2})\ \sqrt{2ADh}$$

If the condition given by (8.13) is satisfied for m = 2, the condition for $m \ge 3$ is automatically satisfied. Therefore the demands for all (n-1) items should be satisfied by item 1 to incur minimum total annual operating cost. Thus the optimal policy is to procure only item 1 for satisfying the demands of all items.

8.6 The Case of n-items with the Different Cost-parameters and Demands:

Now we consider the complete substitution of one item by the others when all parameters are different for each item. In addition, unit substitution cost of item i when it is substituted by item j is also different and is given by V_{ij}.

The total annual cost, when all items are operating under EOQ are given by,

$$TC = \sum_{i=1}^{n} TC_i \tag{8.14}$$

where,

$$TC_i = C_i D_i + \sqrt{2A_i D_i h_i} \quad \text{for } i = 1,2,\ldots,n$$

and

$$EOQ = \sqrt{\frac{2A_i D_i}{h_i}} \tag{8.15}$$

When the demand of item 1 is substituted by item,

$$TC_{ji} = C_i (D_i + D_j) + \sqrt{2A_i (D_i + D_j)\ h_i} + V_{ji}\ D_j \tag{8.16}$$

and the optimal quantity, to be procured, is given by,

$$Q_i^* = \sqrt{\frac{2A_i (D_i + D_j)}{h_i}} \tag{8.17}$$

and reduction in total cost from the total cost under EOQ is given by,

$$RC_{ji} = \sqrt{2A_i D_i h_i} + \sqrt{2A_j D_j h_j} - \sqrt{2A_i (D_i + D_j)\ h_j} + (C_j - C_i - V_{ji})\ D_j \tag{8.18}$$

Let,

- S Set of all substituted items
- S_i Set of items substituted by item i
- V Set of substitutes
- X Set of items selected as substitute

M   Set of items considered as initial substitute

N   Set of all items

T   Set of available substitutes

If item $j \notin S$ is added in the list of S_i then the total annual cost is given by,

$$TC_{ji} = C_i(D_i + \sum_{l \varepsilon \{S_i U\{j\}\}} D_l) + \sum_{l \varepsilon \{S_i U\{j\}\}} V_{li} D_l$$

$$+ \sqrt{2A_i h_i (D_i + \sum_{l \varepsilon \{S_i U\{j\}\}} D_l)} \qquad (8.19)$$

and reduction in total cost is given by,

$$RC_{ji} = (C_j - C_i - V_{ji}) + \sqrt{2A_i h_i (D_i + \sum_{l \varepsilon S_i} D_l)}$$

$$+ \sqrt{2A_j D_j h_j} - \sqrt{2A_i h_i (D_i + \sum_{l \varepsilon \{S_i U\{j\}\}} D_l)}$$

(8.20)

8.6.1 Optimizing Algorithm:

A step-by-step algorithm, for finding the optimal policy for the selection of items among all items, the list of substitutable items and their substitutes is given as follows:

Step 0: (Initialization) Set $M = \phi$, $V = \phi$, $X = \phi$
$Wip = \phi$, $Yip = \phi$, $\forall\, ip \; \varepsilon \; N$

Step 1: For each $i \; \varepsilon \; N$ find total annual cost TC using Eqs. (8.14), set $TC_{min} = TC$

Step 2: Find RC_{ji} using Eq. (8.18) $\forall j \varepsilon N \ \forall i \varepsilon N$,

Step 3: Set $S = \phi$, $T = N$, $S_i = \phi \ \forall i \varepsilon N$

Step 4: If $TC = TC_{min}$, find substituted item j^* and its substitute i^* for $\forall j^* \notin S$ and $\forall i \notin M$ such that total reduction in cost

$$RC_{max} = \max_{\substack{i \varepsilon T \\ j \notin S}} (RC_{ji})$$

Otherwise, find (j^*, i^*) such that

$$RC_{max} = \max \{ \max_{\substack{i \varepsilon T \\ j \notin S}} \{RC_{ji}\}, \max_{\substack{i \varepsilon X \\ j \notin S}} \{RC_{ji}\}\}$$

Step 5: If $RC_{max} \leq 0$, go to Step 7, otherwise set

$TC_{min} = TC_{min} - RC_{max}$,

$S = S \cup \{j^*\}$, $S_i = S_i \cup \{\tilde{j}\}$

$V = V \cup \{1\}$, $X = \{i^*\}$

$T = (N \cap S' \cap V') \cup X$

Step 6: Find RC^*_{ji} using Eq. (8.20) for $j \notin S$ and $i \varepsilon X$, go to Step 4.

Step 7: If $V = \phi$, go to Step 8 otherwise set ip equal to the first element of the set of substitutes, and

minimum total cost $T_{min}(ip) = TC_{min}$

set list substituted items $Y_{ip} = \{S_i | i \in V\}$

and list of substitute $W_{ip} = V$

$V = \phi$, $TC_{min} = TC$

$M = M \cup \{v_1\}$, go to Step 3

Step 8: Find ip, Y_{ip}, W_{ip} $\quad T_{min} = \min_{ip \in m} T_{min}\{ip\}$

stop.

Numerical Example:

For better under standing of the solution procedure, consider four items for the substitution. The relevant data for the problem are given in Table 8.1.

Table 8.1: Basic Data for Numerical Example.

Item j	Procurement cost A_i	Purchase cost C_i	Holding cost h_i	Demand D_i
1	500	43	5.0	1000
2	450	42	6.0	900
3	600	41	5.5	800
4	550	44	5.0	400

$$\{V_{ij}\} = \begin{matrix} - & .9 & 1.0 & .7 \\ .7 & - & 1.1 & .6 \\ .8 & .9 & - & .6 \\ 1.1 & 1.2 & 1.5 & - \end{matrix}$$

Step 0: $N = \{1,2,3,4\}$ $\quad M = \phi$ $\quad V = \phi,$ $\quad X = \phi,$

$W_{ip} = \phi, \quad \forall\, ip \in N$

Step 1: TC_{min} = Rs. 171198

Step 2:

$$\{RC_{ji}\} = \begin{bmatrix} - & 1337.48 & 1992.7 & -402.7 \\ 1628.4 & - & 1062 & -992.4 \\ 1693.8 & 1552.5 & - & -555.14 \\ 1345.37 & 2258.16 & 2866.3 & - \end{bmatrix}$$

Step 3: $S = \emptyset,\ U = \{1,2,3,4\},\ S_1 = S_2 = S_3 = S_4 = \emptyset$

Step 4: $j^* = 4,\quad i^* = 3$

Step 5: Since $RC_{max} > 0$,

$$TC_{min} = 171198.1 - 2866.3 = 168331.7$$

$S = \{u\}\ ,\quad S_3 = \{4\}$

$V = \{3\}\ ,\quad X = \{3\}$

$T = \{1,2,3\}$

Similarly following other steps we have final result as follows:

ip = 3 Yip = {0,0, (1 2,4) 0}

Wip = {3} Total minimum cost = 163798.9

Thus, item 3 is procured to satisfy the demands of all four items. And the total annual operating cost of the system is reduced by the amount Rs. 7359.2 from the total annual cost of four items operating independently.

But above algorithm is based on complete enumeration of the problem. The same work can be extended in the development of the heuristic approach to solve n-item substitution problem to reduce computational effort.

Using the expressions as discussed in (Sec. 8.3 to Sec.8.5) we can reduce the size of the problem. And this feature can be incorporated in the development heuristic approach.

CHAPTER IX

(R,T) MODEL WITH JOINT REPLENISHMENT POLICY

9.1 Statement of Problem:

In an inventory system the demand is an important input parameter. The exact estimation or prediction of demand sometimes is not possible. Therefore, arises the randomness in the demand pattern which significantly complicates the analysis of inventory situation. The randomness in the nature of the demand pattern, introduces the chance of stock out situation for the item. The demand of item under stock out situation can be satisfied by other similar type of item to provide a better service level. Therefore, the concept of the substitution is sometimes useful for the stochastic demand pattern.

We consider here the most widely used operating doctrine for periodic review system, namely order-upto-level (R,T) doctrine, where the inventory position of the item is reviewed at equally spaced interval of length T and an order is placed to bring the inventory position to the level R. We consider the case where all items are reviewed and procured simultaneously. The decision variable then are order-upto-levels for each item and common review period T, under

the situation of substitution, so that the total expected cost of a system is a minimum. For the sake of simplicity we here again consider the case of two items.

9.2 Assumptions:

The following assumptions are made.

1. The cost of making review and order is independent of the values of decision variables.

2. The unit cost of item 1 and item 2 are constant (i.e. independent of the quantity ordered).

3. For each item, shortages occurring in each cycle (review period) are in small quantities. This will be taken to imply that when an order arrives it is almost sufficient to meet any outstanding demands.

4. The demands of items in stock-out conditions are backordered. However, one of the item is substituted by other specifically we take item 1 as the item which is substituted by other item (item 2), if the demand of item 2 is available.

5. The unit cost of backorder is independent of the length of time for which the back-order exists.

6. There is definitely some positive demand for both items between two consecutive reviews so that both items are ordered at each review.

8. The demand of each item is expressed by independent normally distributed random variables with known mean and standard deviation.

9.3 Notations:

The following notations are used for the development of the model.

A The fixed cost of procurement and review

V_1 Unit substitution cost of item 1.

For item i, $i = 1,2$.

C_i Unit cost of item i

h_i Holding cost per unit per unit time for item i

π_i Unit back-order cost of item i

D_i The average demand rate (the expected demand in a year) for item i

R_i Order-up-to-level for item i

μ_i Mean demand during lead time of item i

σ_i Standard deviation of demand during lead time of item i

9.4 Formulation:

Fig. 9.1 depicts the behaviour of inventory levels under (R, T) policy with joint replenishment of both the items. Item 1 and item 2 are reviewed jointly after regular interval of duration T. At each review item 1 and item 2 are ordered-up-to-level R_1 and R_2 respectively. Both items are received after the common constant lead time C. We have shown one of the several possible situations where at point C, item 1 goes to the stock-out position and item 2 has on-hand

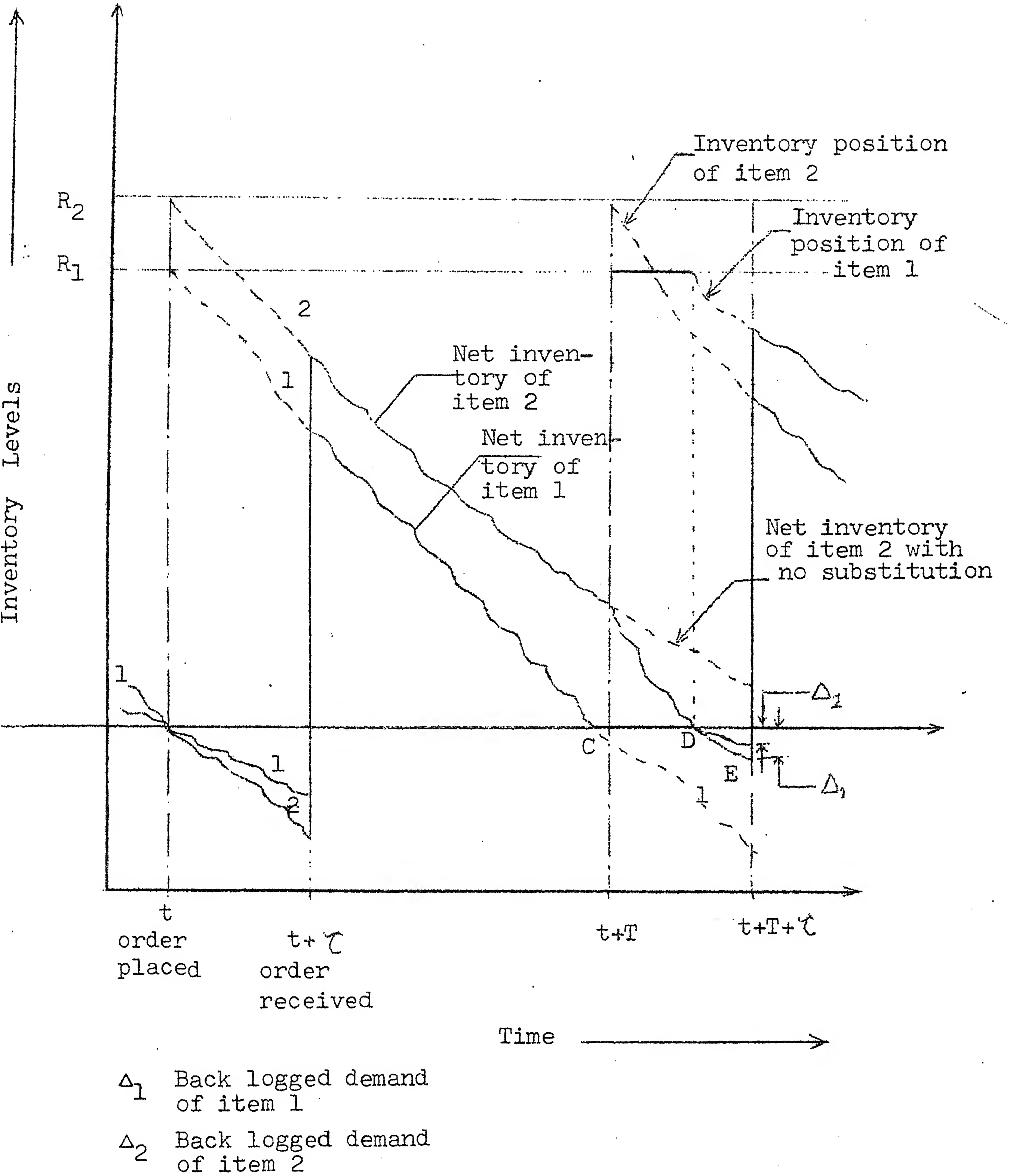

Fig. 9.1: Periodic review (R,T) Policy of two substitutable items.

inventory. Now, item 2 satisfies its own demand as well as the demand of item 1 until it is exhausted at point D. And both items are replenished at point E. Inventory level when there were no substitution, have been shown in dotted line.

Let X_1 and X_2 be independent random variables for demands of item 1 and item 2. And we write $f_{X_1}(\cdot)$ and $F_{X_2}(\cdot)$ to denote the cumulative distribution function, respectively of X_1 and X_2.

There are four mutually exclusive and collectively exhaustive events that can arise for demand x_1 and x_2 of item 1 and item 2. And they are given as follows.

Event	Description	Shortage of Item 1	Shortage of Item 2	Amount of item 1 substituted by item 2
1	$x_1 < R_1$ $x_2 < R_2$	0	0	0
2(a)	$x_1 \geq R_1$; $x_2 < R_2$ and $x_1 + x_2 < R_1+R_2$	0	0	(x_1-R_1)
2(b)	$x_1 \geq R_1$; $x_2 < R_2$ and $x_1 + x_2 > R_1 + R_2$	$(x_1+x_2-(R_1+R_2))$	0	(R_2-x_2)
3	$x_1 \geq R_1$ $x_2 \geq R_2$	x_1-R_1	x_2-R_2	0
4	$x_1 < R_1$ $x_2 \geq R_2$	0	x_2-R_2	0

The substitution of item 1 and item 2 is only possible in events 2(a) and 2(b). The event 2 has following two mutually exclusive condition viz.

2(a) $x_1 \geq R_1$; $x_2 < R_2$

$x_1 + x_2 < R_1 + R_2$

It implies that, during the replenishment cycle the demand of item 2

$$x_2 \leq R_1 + R_2 - x_1$$

and $(x_1 - R_1)$ units of item 1 are substituted by item 2. Therefore expected units of item 1 substituted by item 2 per replenishment cycle with this condition are given by

$$ES^a_{12} = \int_{R_1}^{\infty} (x_1 - R_1) \int_{-\infty}^{R_1+R_2-x_1} f_{X_1}(x_2)\, dx_2\, f_{X_2}(x_1) dx_1 \quad (9.1)$$

2(b) $x_1 + x_2 > R_1 + R_2$

$x_1 \geq R_1$ and

$x_2 < R_2$

It implies that $R_2 > x_2 \geq R_1 + R_2 - x_1$ and the demand equal to $(R_2 - x_2)$ units of item 1 is satisfied by item 2. Therefore, the expected units of item 1 substituted by item 2 per replenishment cycle with condition 2(b) are given by,

$$ES_{12}^{b} = \int_{R_1}^{\infty} \int_{R_1+R_2-x_1}^{R_2} (R_2-x_2)\, f_{X_2}(x_2)\, dx_2\, f_{X_1}(x_1)\, dx_1 \tag{9.2}$$

Therefore the expected substitution of item 1 by item 2 per replenishment is given by,

$$\begin{aligned} ES_{12} &= ES_{12}^{a} + ES_{12}^{b} \\ &= \int_{R_1}^{\infty} (x_1-R_1) \int_{-\infty}^{R_1+R_2-x_1} f_{X_2}(x_2)\, dx_2\, f_{X_1}(x_1)\, dx_1 \\ &\quad + \int_{R_1}^{\infty} \int_{R_1+R_2-x_1}^{R_2} (R_2-x_2)\, f_{X_2}(x_2)\, dx_2\, f_{X_1}(x_1) dx_1 \end{aligned} \tag{9.3}$$

And expected backorders of item 1 per replenishment cycle are given by

$$\begin{aligned} B^{(1)}(R_1,R_2) &= \int_{R_1}^{\infty} (x_1-R_1) \int_{R_2}^{\infty} f_{X_2}(x_2)\, dx_2\, f_{X_1}(x_1) dx_1 \\ &\quad + \int_{R_1}^{\infty} \int_{R_1+R_2-x_1}^{R_2} (x_1+x_2 - R_1-R_2)\, x \\ &\quad x\ f_{X_2}(x_2)\, dx_2\ f_{X_1}(x_1)\, dx_1 \end{aligned} \tag{9.4}$$

Similarly expected backorders of item 2 per replenishment cycle is given by,

$$B^{2}(R_1,R_2) = \int_{R_2}^{\infty} (x_2-R_2)\, f_{X_2}(x_2)\, dx_2 \tag{9.5}$$

The various relevant costs are given as follows:

A. Average annual fixed ordering cost $= A/T$ (9.6)

B. Average annual carrying cost.

Expected net inventory prior to receive of order:

$$\text{For item 1} = R_1 - \mu_1 - D_1 T + ES_{12} \quad (9.7)$$

$$\text{For item 2} = R_2 - \mu_2 - D_2 T - ES_{12} \quad (9.8)$$

Expected net inventory immediately after receive of order.

$$\text{For item 1} = R_1 - \mu_1 + ES_{12} \quad (9.9)$$

$$\text{For item 2} = R_2 - \mu_2 - ES_{12} \quad (9.10)$$

Therefore, expected net inventory,

$$\text{For item 1} = R_1 - \mu_1 - \frac{1}{2} D_1 T + ES_{12} \quad (9.11)$$

$$\text{For item 2} = R_2 - \mu_2 - \frac{1}{2} D_2 T - ES_{12} \quad (9.12)$$

Therefore, average annual carrying cost is,

$$C_H = h_1(R_1 - \mu_1 - \frac{1}{2} D_1 T + ES_{12}) + h_2 (R_2 - \mu_2 - \frac{1}{2} D_2 T - ES_{12}) \quad (9.13)$$

C. Average annual back-order cost

$$C_B = \frac{\pi_1}{T} B^{(1)}(R_1, R_2) + \frac{\pi_2}{T} B^{(2)} (R_1, R_2) \quad (9.14)$$

D. Average annual purchase cost

For item 1 $= \frac{1}{T} C_1$ [$D_1 T$ - Expected substitution of item 1 by item 2 during replenishment cycle]

$$= C_1 D_1 - \frac{C_1}{T} ES_{12} \quad (9.15)$$

For item 2 $= \frac{1}{T} C_2$ [D_2T + Expected substitution of item 1 by item 2 during replenishment cycle]

$$= C_2D_2 + \frac{C_2}{T} ES_{12} \tag{9.16}$$

E. Average annual substitution cost of item 1

$$C_S = V_1 \frac{ES_{12}}{T} \tag{9.17}$$

Therefore expected annual totalr rcost which is the sum of carrying, backordering, substitution and ordering cost is given by

$$TC = h_1R_1 + h_2R_2 + (h_1-h_2)\ ES_{12} + \frac{\pi_1}{T} B^{(1)}\ (R_1,R_2) + \frac{\pi_2}{T} B^{(2)}\ (R_1,R_2) + \frac{(C_2-C_1+V_1)}{T} ES_{12} + K_C(T) \tag{9.18}$$

where $KC(T) = C_1D_1 + C_2D_2 - h_1\mu_1 - h_2\ \mu_2 - \frac{1}{2} h_1D_1T - \frac{1}{2} h_1D_1T - \frac{1}{2} h_2D_2T + \frac{A}{T}$

The average annual total cost given by Eq. (9.18) is function of decision variables R_1, R_2 and T. For item i, i = 1,2 we express order-up-to-level R_i as

$$R_i = \mu_i + k_i\ \sigma_i \tag{9.19}$$

where K_i is policy variable for item i.

Using the transformation,

$$Z_i = \frac{x_i - \mu_i}{\sigma_i}$$

for Eqs. (9.3) - (9.5) we get,

$$ES_{12} = \int_{k_1}^{\infty} \sigma_1(z_1-k_1) \int_{-\infty}^{-\sigma_1/\sigma_2 (z_1-k_1)+k_2} \int_{-\infty}$$

$$+ \int_{k_1}^{\infty} \int_{-\frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2}^{k_2} \sigma_2(k_2-z_2)\, \phi(z_2)dz_2\, \phi(z_1)dz_1$$

$$B^{(1)}(k_1,k_2) = \int_{k_1}^{\infty} \sigma_1(z_1-k_1)\, \Phi\left(-\frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2\right) \phi(z_1)dz_1$$

$$+ \int_{k_1}^{\infty} \int_{-\frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2}^{k_2} \sigma_2(z_2-k_2)\, \phi(z_2)\, dz_2\, \phi(z_1)dz_1$$

and,

$$B^{(2)}(k_1,k_2) = \int_{k_2}^{\infty} \sigma_2(z_2-k_2)\, \phi(z_2)\, dz_2$$

where $\phi(.)$ standard normal density function and

$\Phi(.)$ cumulative distribution function.

Therefore, average annual total cost can be rewritten as

$$TC = h_1k_1\sigma_1 + h_2k_2\sigma_2 + (h_1-h_2)\, ES_{12}$$

$$+ \frac{1}{T}\pi_1 B^{(1)}(k_1,k_2) + \frac{1}{T}\pi_2 B^{(2)}(k_1,k_2)$$

$$+ \frac{(C_2-C_1+V_1)}{T} ES_{12} + KC(t) \qquad (9.22)$$

where,

$$KC(t) = C_1D_1 + C_2D_2 - \frac{1}{2} h_1D_1T - \frac{1}{2} h_2D_2T + A/T$$

For a given T, setting $\frac{\partial TC}{\partial k_i} = 0$ give best k_i for $i = 1,2$ for the particular set of T.

$$\frac{\partial TC}{\partial k_1} = h_1\sigma_1 + \{(h_1-h_2) + \frac{(C_2-C_1+V_1)}{T}\} \frac{\partial ES_{12}}{\partial k_1}$$

$$+ \frac{\pi_1}{T} \frac{\partial B^{(1)}}{\partial k_1} (k_1, k_1)$$

$$+ \frac{\pi_2}{T} \frac{\partial B^{(2)}}{\partial k_1} (k_1, k_1) = 0 \qquad (9.23)$$

$$\frac{\partial TC}{\partial k_2} = h_2\sigma_2 + [h_1-h_2 + (\frac{C_2-C_1+V_1}{T})] \frac{\partial ES_{12}}{\partial k_2}$$

$$+ \frac{\pi_1}{T} \frac{\partial B^{(1)}}{\partial k_2} (k_1, k_2)$$

$$+ \frac{\pi_2}{T} \frac{\partial B^{(2)}}{\partial k_2} (k_1, k_2) = 0 \qquad (9.24)$$

By applying Leibzinth rule, we get (the details are shown in Appendix A)

$$\frac{\partial ES_{12}}{\partial k_1} = -\sigma_1 \int_{k_1}^{\infty} \Phi\left(-\frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2\right) \phi(z_1)dz_1 \qquad (9.25)$$

$$\frac{\partial ES_{12}}{\partial k_2} = \sigma_2 \Phi(k_2) [1-\Phi(k_1)] - \sigma_2 \int_{k_1}^{\infty} \Phi\left(-\frac{\sigma_1}{\sigma_2}(z_1-k_1)\right.$$

$$\left. + k_2\right) \phi(z_1)\, dz_1) \qquad (9.26)$$

$$\frac{\partial B^{(1)}(k_1,k_2)}{\partial \quad k_1} = -\sigma_1 [1-\Phi(k_1)] + \sigma_1 \int_{k_1}^{\infty} \Phi\left(-\frac{\sigma_1}{\sigma_2}(z_1-k_1)\right.$$

$$\left. + k_2\right) \phi(z_1)\, dz_1 \tag{9.27}$$

$$\frac{\partial B^{(1)}(k_1,k_2)}{\partial \quad k_2} = -\sigma_2 \Phi(k_2)\, [1-\Phi(k_1)] + \sigma_2 \int_{k_1}^{\infty} \Phi\left(-\frac{\sigma_1}{\sigma_2}(z_1\right.$$

$$\left. -k_1) + k_2\right) \phi\, (z_1)\, dz_1 \tag{9.28}$$

$$\frac{\partial B^{(2)}(k_1,k_2)}{\partial \quad k_2} = -\sigma_2 [1-\Phi(k_2)] \tag{9.29}$$

From Eqs. (9.23), (9.25) and (9.27) we get,

$$\int_{k_1}^{\infty} \Phi\left[-\frac{\sigma_1}{\sigma_2}(z_1-k_1) + k_2\right] \phi(z_1) dz_1 = \frac{-h_1 + \frac{\pi_1}{T}(1-\Phi(k_1))}{\frac{\pi_1}{T} - (h_1-h_2+\frac{C_2-C_1+V_1}{T})} \tag{9.30}$$

From Eqs. (9.24), (9.26), (9.28) and (9.29), we get,

$$\int_{k_1}^{\infty} \Phi\left(-\frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2\right) \phi(z_1) dz_1 = \frac{\left[-h_2 + \left[\frac{\pi_1}{T} - (h_1-h_2 + \frac{C_2-C_1+V_1}{T})\right] \Phi(k_2)(1-\Phi(k_1)) + \frac{\pi_2}{T}[1-\Phi(k_2)]\right]}{\left[\frac{\pi_1}{T} - (h_1-h_2+\frac{C_2-C_1+V_1}{T})\right]} \tag{9.31}$$

Solving Eqs. (9.30) and (9.31) we get the optimal value of k_1 in terms of k_2 and is given by,

$$\Phi(k_1) = \frac{(h_1-h_2) + \frac{\pi_2-\pi_1}{T} + \left[\frac{\pi_1-\pi_2}{T} - \{h_1-h_2+ \frac{C_2-C_1+V_1}{T}\}\right] \Phi(k_2)}{\left[\frac{\pi_1}{T} - \Phi(k_2) \left\{ \frac{\pi_1}{T} - (h_1-h_2+ \frac{C_2-C_1+V_1}{T} \right\}\right]} \tag{9.32}$$

The steps involved in solution procedure are as follows.

<u>Step 1:</u> Choose the value of periodic review period T.

<u>Step 2:</u> Calculate k_2 from the following expression for no substitution,

$$\Phi^c(k_2) = \frac{h_2 T}{\pi_2}$$

<u>Step 3:</u> Calculate optimal k_1 from Eq. (9.32).

<u>Step 4:</u> Calculate total cost given by Eq. (9.22) using some numerical technique for the integration.

<u>Step 5:</u> Change the value of T and Go to Step 2 till we achieve some minimum value of total annual cost.

Since the normal density function complicate the integration and exact value of the integration is not possible we can used some numerical technique to calculate average annual cost. Alternately, we can simplify the procedure by approximating the standard normal density functions to a suitable function so that we can find out the exact value of the

integration and thus we can obtain the optimal value of decision variables k_1, k_2 and T for minimum cost.

The approach discussed previously are based on mathematical model. But we can use simulation instead, for developing decision rules, where due to increase complexity the mathematical model can not be easily solved.

CHAPTER X

(Q, r, T) MODEL WITH SUBSTITUTION

10.1 Statement of Problem:

In previous chapter, we have discussed (R, T) model for two items with substitution. In (R,T) model at every review an order is placed for the item to bring the inventory position to a level R. In this chapter, we shall discuss (Q, r, T) model in which at every review at the interval T, an order of size Q is placed only if the inventory position of the item falls below a predefined level (re-order level) r.

For a two-item inventory system with stochastic demands we consider the coordinated replenishment policy. In this policy whenever one item is ordered, other item can also be replenished along with it in order to save on major ordering (set-up) cost that probably would have occurred in near future if these items were ordered separately. For each item we determine a level, can order level . Whenever an order is being placed for an item whose inventory position has fallen below its reorder level, other item whose inventory position is below can order level but above its reorder level is also ordered.

We consider the case where both items can substitute each other in stock-out situation. The level of substitution is at most equal to a given proportion (substitution factor) of the total shortages. Shortages not substituted are backlogged.

The objective is to decide quantities to be ordered, reorder level and can order levels for each item and review period such that the expected total cost per unit time is minimized.

10.2 Assumptions:

1. The demand for each item is stochastics and the parameters of its distribution are known.

2. There is a common lead time for both items and it is constant.

3. All the cost parameters are independent of time and there is no quantity discount.

4. The customers are arriving in random manner and the quantities demanded are also random.

10.3 Notations:

Following notations are used for the description of simulation model for present inventory system.

A	Major ordering cost or setup cost
T	Review period
τ	Lead time

For item i, $i = 1,2$.

a_i	Minor ordering cost of item i
h_i	Holding cost of item i charged Rs/unit/unit time
$\tilde{\pi}_i$	Back-ordering cost charged Rs/unit/unit time
v_i	Substitution cost of item i in Rs/unit time
α_i	Substitution factor of item 1
λ_i	Mean arrival rate of customer i
p_i	Probability for a demand of item i by customer i
$D_M(i)$	Minimum demand level of item i
Q_i	Quantities of item i to be procured
O_i	On hand inventory of item i
B_i	Backlogged units of item i
i_i	Inventory position of item i at present review
r_i	Reorder level of item i
Cr_i	Can order level of item i
$SQTY_i$	Substituted units of item i by other duration δ_i
HC_i	Holding cost of item i
BC_i	Backlogging cost of item i for the duration δ_i
SC_i	Substitution cost of item i
ASQ_i	Average substituted quantities of item i by the other
$AHCC_i$	Average holding cost of item i
ABC_i	Average Backlogging cost of item i
TA	Expected average total cost

10.4 System Description:

The inclusion of can order level, substitutability and randomness of several variables (customer arrival and demand size) make it very difficult to represent the system mathematically. With this in view, we solve the problem using simulation technique. For the purpose of simulation model, we make the following assumptions.

1. Fixed ordering cost are paid at the time of review if the item is ordered.

2. The purchase cost of an item is paid when the item is received.

The description of the simulation model which uses event scheduling approach is as follows:

10.4.1 Entities:

The various entities in the system are:

a. Items:

The two items are denoted as item 1 and item 2.

b. Customers:

The customers demanding for units of item 1 and item 2 are labelled as customer 1 and customer 2, respectively.

10.4.2 Events:

Following are the various events which occur in the system.

A. Customers Arrival:

The arrival of the customer i changes the state of inventory position, on hand inventory (negative on hand inventory indicates the backlogging) of item i. On occurrence of this event the various costs are updated as follows:

a) If an item i is held for the duration δ_i, the holding cost is given by,

$$HC_i = h_i O_i \delta_i \tag{10.1}$$

b) If an item i is backlogged for the duration $_i$, the backlogging cost is given by,

$$BC_i = \tilde{\pi}_i \; B_i \delta_i \tag{10.2}$$

c) If $SQTY_i$ units of item i are substituted by the other item, the substitution cost is given by,

$$SC_i = v_i \; SQTY_i \tag{10.3}$$

Let on arrival of customer i, QTY_i units of item 1 be demanded by him. Therefore, on occurrence of demand for item i, we check the following condition for updating the record.

a) If $O_i > QTY_i$, the quantity demanded is satisfied. The holding cost is calculated from Eq. (10.1) and inventory position, on hand inventory is updated.

b) If $O_i < QTY_i$ and O_i is greater than zero, the portion of the demand equal to O_i is satisfied and holding cost of item i is calculated, using Eq. (10,1).

The maximum allowable substitution (MAS) of the unsatisfied demand is calculated by multiplying the total number of units of unsatisfied demand with the substitution factor and then we check following two conditions.

1) If on hand inventory of second item is greater than or equal to MAS, then MAS units of the demand of the first item are satisfied using on hand inventory of item 2 and balance is backlogged, holding cost of the second item is calculated using Eq.(10.1), and backlogging and substitution cost of first item are calculated using Eq. (10.2) and (10.3), respectively. On hand inventory, inventory position of both the items are updated.

2) If on hand inventory of second item is less than MAS then units equal to on hand inventory of second item are used to satisfy the demand of first item, and balance of the demand of first item is backlogged. And various relevant costs are calculated by using Eqs. (10.1) to (10.3). Inventory position, on hand inventory of the both items are updated.

Cummulative total cost is updated by adding various cost calculated above. Everytime, when an arrival of customer i occurs we schedule the next arrival of the customer.

B. <u>Arrival of Procurement</u>:

When the supply of the item(s) from external source arrives, the on hand inventory, inventory position and

backorders of the corresponding item(s) are updated and purchase cost of item(s) is added to the total cost.

C. Review:

At the beginning of each review period we check inventory position of each item. All the possible situations and the corresponding actions are given below:

a) $i_1 \leq r_1$ and $i_2 \leq Cr_2$	Order for both the items is placed.
b) $i_1 \leq r_1$ and $i_2 > Cr_2$	Order for item 1 is placed
c) $i_1 \leq Cr_1$ and $i_2 \leq r_2$	Order for both the items is placed
d) $i_2 \leq r_2$ and $i_1 > Cr_1$	Order for item 2 is placed.
e) $i_1 > r_1$ and $i_2 > r_2$	No order is placed.

According to above situations the inventory position, on order quantities of both the items are suitably updated and review cost ordering cost are added to the total cost. The time of next review is scheduled every time when a reviewing event takes place. Whenever an order is placed the arrival of its procurement is also scheduled.

Priority Order:

When two or more events are occurring simultaneously, the events are selected according to the following priority rule.

Table 10.5: List of variables.

```
H(I):=HOLDING COST PER UNIT ITEM PER UNIT TIME OF ITEM  "I".
B(I):=BACKLOGGING COST PER UNIT ITEM PER UNIT TIME OF ITEM "I".
BA(I):=BACKLOGGING COST PER UNIT SHORTAGE OF ITEM "I".
A(I):=FIXED MINOR SETUP COST OF ITEM "I".
V(I):=SUBSTITUTION COST PER UNIT SUBSTITUTION OF ITEM "I".
C(I):=THE PURCHASE COST OF ITEM "I".
AL(I):=ARRIVAL RATE OF CUSTOMER FOR THE DEMAND OF ITEM "I".
ALPH(I):=SUBSTITUTION FACTOR (THE BACLOGGED DEMANDS OF ITEM "I"
        IS SATISFIED BY OTHER ITEM)
PAL(I):=PROBABILITY OF DEMAND OF THE ITEM "I" (FOR GEOMETRIC
        DISTRIBUTION OF DEMAND OF ITEM "I" BY THE CUSTOMER )
T:=PERIODIC REVIEW PERIOD.
NLD(I):=NEXT OCCURENCE OF LEAD PERIOD
NRP:=NEXT OCCURENCE OF REVIEW PERIOD.
A:=FIXED SET-UP COST
R:=REVIEW COST.
OHP(I):=ON HAND INVENTORY OF ITEM "I".
IOP(I):=INVENTORY POSITION OF ITEM "I".
BO(I):=BACK-ORDERS OF ITEM "I".
SQ(2,1):=SUBSTITUTED QUANTITY OF ITEM2 BY ITEM1
SQ(1,2):=SUBSTITUDED QUANTITY OF ITEM1 BY ITEM2
BC(I):=BACKODERS COST OF ITEM "I".
HC(I):=HOLDING COST OF ITEM "I".
SC(I):=SUBSTITUTION COST OF ITEM "I".
TPC(I):=TOTAL PURCHASE COST OF ITEM "I".
ABC(I):=AVERAGE BACKODERS COST OF ITEM "I".
ASC(I):=AVERAGE SUBSTITUTION COST OF ITEM "I".
AHCC(I):=AVERAGE HOLDING COST OF ITEM "I".
HCC(I):=TOTAL HOLDING COST OF ITEM "I".
TAU:=CONSTANT LEAD TIME FOR RECIEVING THE SUPPLY
R1:=REORDER LEVEL OF ITEM1
R2:=REORDER LEVEL OF ITEM2
CR1:=CAN ORDER LEVEL OF ITEM1
CR2:=CAN ORDER LEVEL OF ITEM2
Q1:=QUANTITY TO BE ORDER OF ITEM "I".
Q2:=QUANTITY TO BE ORDER OF ITEM "I".
CYT:=PRESENT CYCLE TIME.
SQTY:=SUBSTITUTED QUANTITY OF ONE ITEM BY OTHER
PT(I):=PREVIOUS DEMAND OCCURENCE TIME OF ITEM " I ".
CT1:=CYCLE TIME OF ITEM 1.
CT2:=CYCLE TIME OF ITEM 2.
```

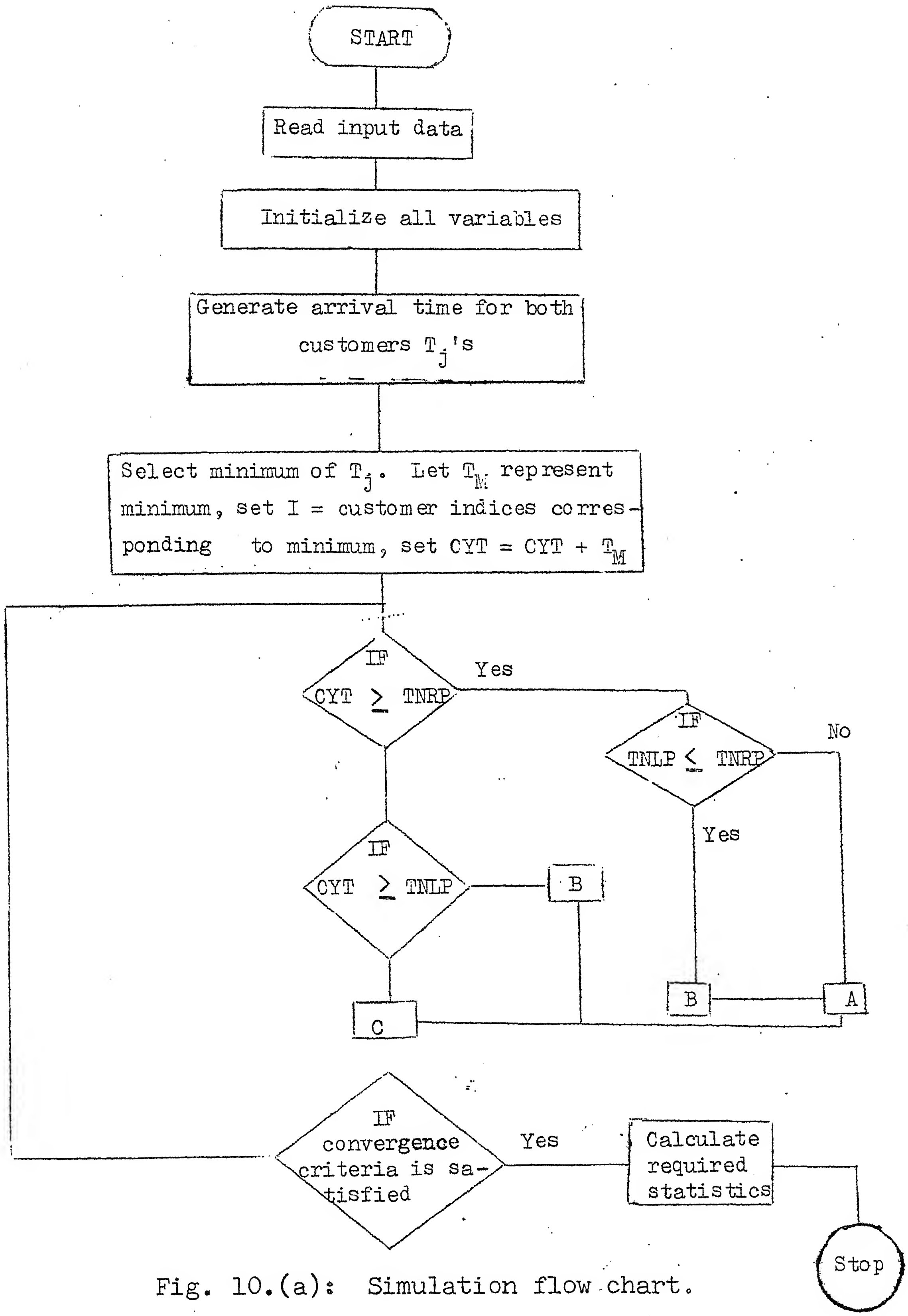

Fig. 10.(a): Simulation flow chart.

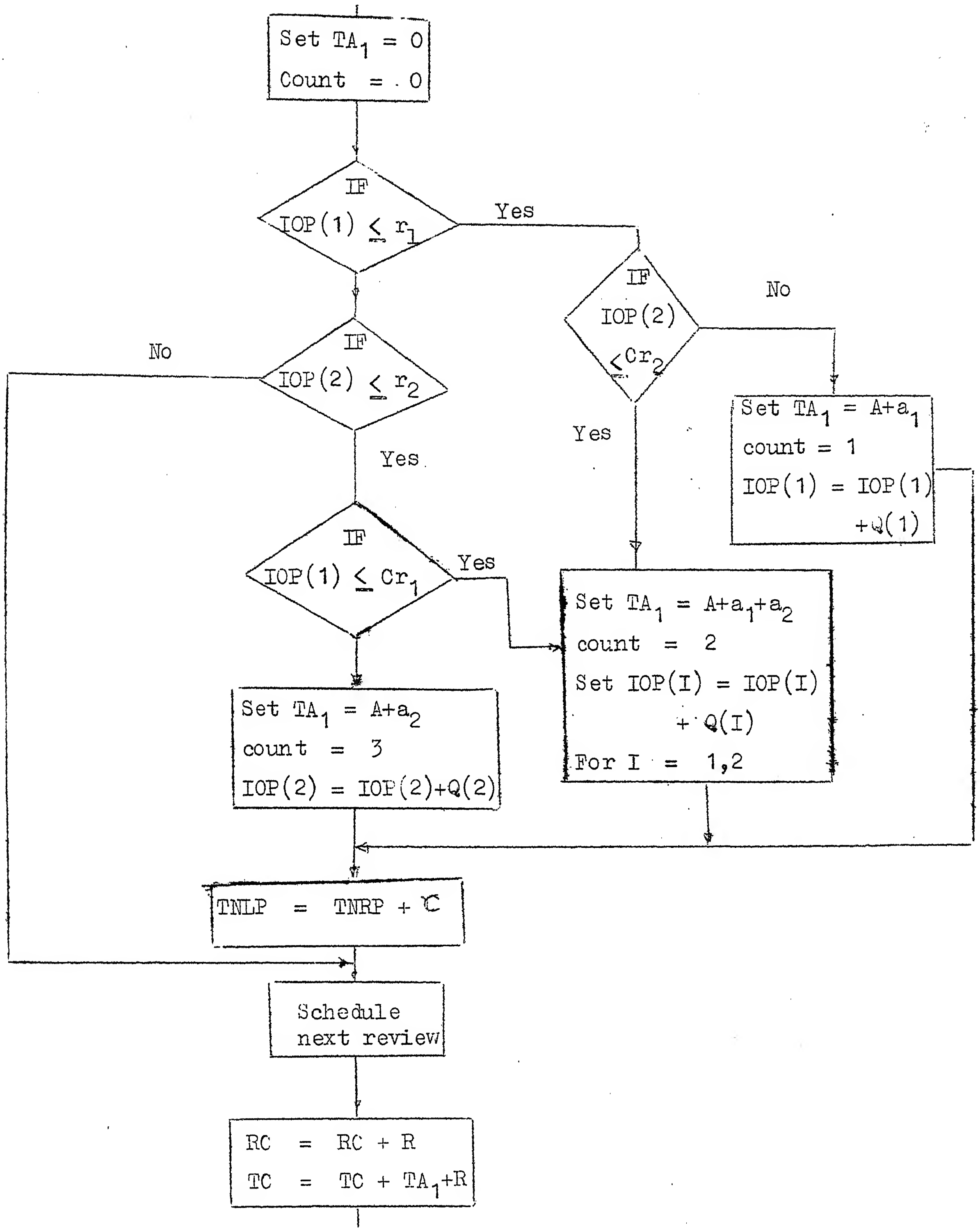

Fig. 10.1(c): Block B of Fig. 10.(a).

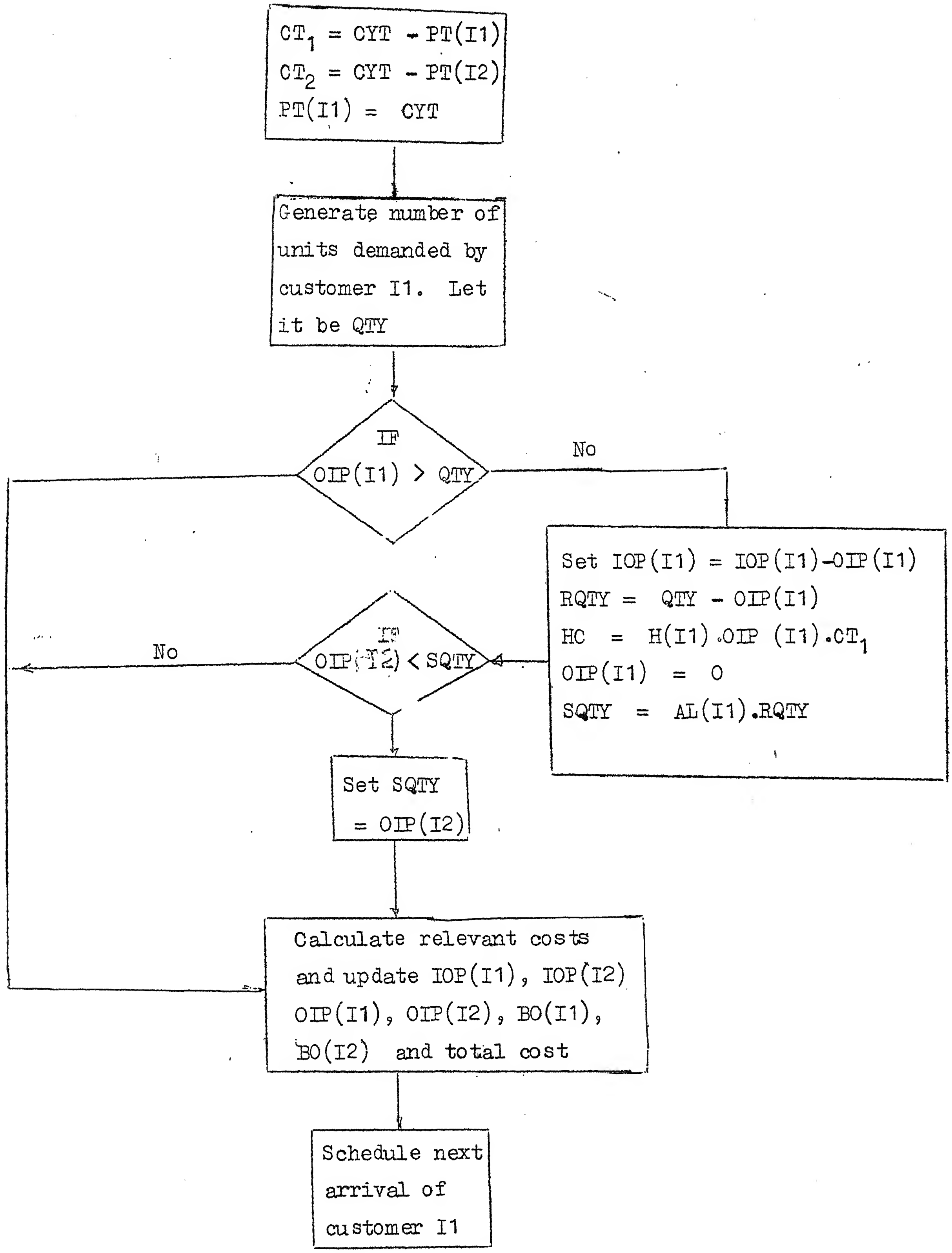

Fig. 10.1(d): Block C in 10.1(a), Customer arrival.

1. Arrival of procurement
2. Review
3. Customer arrival

Whenever an event ends, the next event for the system is selected. After certain number of review periods, the termination criteria is checked and if it gets satisfied the simulation is stopped. Flow chart for the problem is given in Figs. 10.1(a) - (d). The list of variables used in simulation is shown in Table 10.5.

10.5 Performance Measures and General Constraints:

Following measures can be of importance for evaluating the performance of the system under study.

1. Expected average total cost per unit time
2. Service level
3. Total investment in certain span of time
4. Average turn-over data

Out of all above mentioned measures, it is the minimization of expected total cost per unit time which has, commonly been, considered as the objective for such problems, Under certain special circumstances, like budget constraint instead of minimizing the expected total average cost per unit time it is usually preferred to minimize total cost during certain span of time as dictated by the constraints. Besides these two, maximization of service level and/or

turnover rate can also be considered as objectives for deciding the inventory policy with substitution.

Any measure of performance either by itself or along with other measures of performance can be treated as objective function or it can be included in the constraint set.

In the constraint set we may include maximum number of reviews in certain span of time, maximum number of units backordered, allowable level of fraction of average demand backlogged, maximum limit on total investment in certain span of time (minimum allowable service level), maximum number of units of an item used as a substitute, maximum number of orders placed during certain span of time and so on.

For our present purpose we are considering the expected average total cost per unit time as objective function with no constraints.

10.6 Termination Criteria:

Sometimes, in simulation studies, the termination criterion is either expressed in terms of elapsed number of simulated time units or in terms of number of customers. In most of the cases, some statistical termination criterion is also employed to ascertain the results obtained after

simulation with certain degree of confidence in them, to make sure that the system has attained the steady state. The statistical controls can be employed accurately only when the exact distribution pattern and parameters are known for all the variables of which the statistical stability is being examined for termination of simulation.

For our present simulation purpose we have fixed the maximum limit on simulation duration, as one of the Termination criteria. Since our main objective is to minimize the expected average total cost, we examine the statistical stability of this. Since exact distribution for the expected total cost per unit time is not known, it is approximated to a normal distribution. The parameters of the distribution all estimated from the data generated by the simulation itself. When one of the two termination criteria gets satisfied, the simulation is terminated.

10.7 Data Collection:

At the end of simulation we would like to collect information about expected total cost and its various components, total duration of simulation. The data about the expected average substitution of one item by the other and probabilities of ordering item 1 and/or item 2 at a review are also collected. These probabilities can be used to calculate average review cost, expected orders of both item, when the model is approximated as the deterministic one.

10.8 Solution Procedure:

There are seven decision variables in the system. In order to obtain the optimal ordering policy a very large number of combination of these variables will have to be tried. This will require enormous computational efforts. To reduce this computational effort, we propose to cut:-down the total number of such combinations. To do so, we solve an approximate deterministic coordinated replenishment problem with shortages allowed to backlog for both the items operating independently. The optimal ordering quantities (Q_1, Q_2) and the reviewing period T obtained for this problem can be used as an initial guess for the decision variables (Q_1, Q_2) for present system. Naddor approximations [8] can now be applied to obtain reordering levels for both items. Can order levels for both the items can be set to 20 to 30 percent of ordering quantities.

At any instant of time, out of seven decision variables, we make unidirectional search along one of these keeping all other fixed. Unimodality of the objective function is assumed. For each set of decision variable, the objective function is now obtained using simulation. The decision variable which was sofar, being explores, can now be set at the optimal value obtained after this exploration and the same procedure will now be repeated one by one for all the decision variables, in the following order,

$T, Cr_2, Cr_1, r_2, r_1, Q_2, Q_1$

For each of the decision variables, the unidirectional search can be divided in two steps. Firstly, we make a block search to obtain the initial interval of the uncertainty by selecting a suitable step length. This interval then can further be explored, using Golden section method for obtaining the approximate optimal value for the decision variables.

The solution obtained, after making above mentioned unidirectional search for all the decision variables can be claimed to be an approximate optimal solution.

10.9 Numerical Example:

To illustrate the solution procedure we take a numerical example whose parameters are given in Table 10.1.

Since the distribution for the demand that we are considering is stuttering poisson. It is very difficult to obtain initial guess for finding the variables. We, therefore, select the initial values for the variables as,

$Q_1 = 95 \qquad Q_2 = 85 \qquad r_1 = 12 \qquad r_2 = 13$

$Cr_1 = 42, \qquad Cr_2 = 43 \qquad T = .5$

The result for the block search and golden section search for variable T keeping all other variable constant is shown in Table 10.2. Final interval of uncertainty for variable T using golden section search technique is one day.

TABLE 10.1: INPUT PARAMETERS FOR THE EXAMPLE IN CHAPTER 10.

TIME LIMIT	5000.00
LEAD TIME(CONSTANT)	0.10
TERMINATION FACTOR	0.05
MAJOR FIXED(SET-UP) COST	500.00
PERIODIC REVIEW COST	10.00

ITEM NO.	HOLDING COST	BACKLOGGING COST	COST /UNIT	FIXED COST	SUBSTITUTION COST PER UNIT.	UNIT COST	ARRIVAL RATE	SUBSTITUTION FACTOR	PROBABILITY OF DEMAND
1	2.50	8.00	2.00	50.00	0.90	25.00	0.0500	0.90	0.60
2	3.20	9.10	2.50	60.00	0.80	23.00	0.0400	0.80	0.50

CUSTOMER ARRIVAL : EXPONENTIAL DISTRIBUTION.
DEMAND : GEOMETRIC DISTRIBUTION (MINIMUM 5 UNITS).
RESULTANT DISTRIBUTION FOR DEMAND IS STUTTERING POISSON.

TABLE 1?.2: BLOCK AND GOLDEN SECTION SEARCH FOR A VARIABLE (T).

R1	R2	CR1	CR2	Q1	Q2	T	ETOTANC
22	23	42	43	95	85	.50	8104.
22	23	42	43	95	85	.60	8113.
22	23	42	43	95	85	.70	21547.
22	23	42	43	95	85	.40	8114.
22	23	42	43	95	85	.48	8102.
22	23	42	43	95	85	.52	8093.
22	23	42	43	95	85	.55	8053.
22	23	42	43	95	85	.57	8058.
22	23	42	43	95	85	.54	8062.
22	23	42	43	95	85	.53	8066.
22	23	42	43	95	85	.55	8053.
22	23	42	43	95	85	.55	8056.

TABLE 10.3: APPROXIMATE OPTIMAL SOLUTION WITH SUBSTITUTION.

S1	125.00
S2	135.00
R1	24.00
R2	15.00
CR1	61.00
CR2	43.00
T (IN DAYS)	200.00
ELAPSED NUMBER OF SIMULATED TIME UNITS	384.00
EXPECTED ANNUAL REVIEW COST	18.20
EXPECTED ANNUAL FIXED COST	590.00
EXPECTED ANNUAL PURCHASE COST OF ITEM1	3402.00
EXPECTED ANNUAL PURCHASE COST OF ITEM2	3324.00
EXPECTED ANNUAL HOLDING COST OF ITEM1	141.00
EXPECTED ANNUAL HOLDING COST OF ITEM2	152.00
EXPECTED ANNUAL BACKORDERING COST FOR ITEM 1	20.00
EXPECTED ANNUAL BACKORDERING COST FOR ITEM 2	33.00
EXPECTED ANNUAL SUBSTITUTION COST FOR ITEM 1	4.80
EXPECTED ANNUAL SUBSTITUTION COST FOR ITEM 2	9.00
EXPECTED ANNUAL TOTAL COST	7803.29
EXPECTED ANNUAL SUBSTITUTION OF ITEM 1 BY ITEM 2	4.70
EXPECTED ANNUAL SUBSTITUTION OF ITEM 2 BY ITEM 1	11.30
AT THE TIME OF REVIEW PROBABILITY OF ORDERING ITEM 1	.04
PROBABILITY OF ORDERING ITEM 2	.03
PROBABILITY OF ORDERING BOTH	.56

TABLE 10.4: APPROXIMATE OPTIMAL SOLUTION WITH NO SUBSTITUTION.

[illegible]	121.00
[illegible]	110.00
[illegible]	10.00
[illegible]	23.00
CP1	40.00
CP2	71.00
T (IN DAYS)	182.00
ELAPSED NUMBER OF SIMULATED TIME UNITS	350.00
EXPECTED ANNUAL REVIEW COST	20.00
EXPECTED ANNUAL FIXED COST	847.00
EXPECTED ANNUAL PURCHASE COST OF ITEM1	3249.00
EXPECTED ANNUAL PURCHASE COST OF ITEM2	3462.00
EXPECTED ANNUAL HOLDING COST OF ITEM1	124.00
EXPECTED ANNUAL HOLDING COST OF ITEM2	153.00
EXPECTED ANNUAL BACKORDERING COST FOR ITEM 1	32.00
EXPECTED ANNUAL BACKORDERING COST FOR ITEM 2	42.00
EXPECTED ANNUAL SUBSTITUTION COST FOR ITEM 1	00.00
EXPECTED ANNUAL SUBSTITUTION COST FOR ITEM 2	00.00
EXPECTED ANNUAL TOTAL COST	7929.20
EXPECTED ANNUAL SUBSTITUTION OF ITEM 1 FOR ITEM 2	0.00
EXPECTED ANNUAL SUBSTITUTION OF ITEM 2 FOR ITEM 1	0.00
AT THE TIME OF REVIEW PROBABILITY OF ORDERING ITEM 1	.08
PROBABILITY OF ORDERING ITEM 2	.17
PROBABILITY OF ORDERING BOTH	.51

Now value of T is set equal to the value which corresponds to minimum objective function in above search. This process is now repeated for all other variables. The final results are shown in Table 10.3. We resolved the problem for no substitution case and final results are shown in Table 10.4.

10.9.1 Comment:

The results obtained above show a substantial improvement in total cost for the case when substitution is permitted as compare to the case when no substitution is allowed.

At the same time, the operating levels of inventory system for the case of substitution is quite different from that of no substitution.

10.10 Scope of Future Studies:

We can improve further, the present solution procedure by scaning first time whole cycle from T to Q_1 with coarser final interval of uncertainty and repeating the cycle from T to Q_1 second time with finner final interval of uncertainty. The initial interval of uncertainty for the variables in second cycle are coarser final interval of uncertainty obtained from previous cycle. Once second scan is over, it can be expected that the values of the variables, so obtained, are very close to actual optimum values.

We can consider the inventory system with probabilistic lead time distribution.

In Sec. 10.5, we have mentioned four possible performance for our system. One can think of many more similar performance measures. For our present study we concentrated only on the average annual total cost, but combination of these performance measures can be used as an objective function and we can take one or more as a constant on the system.

CHAPTER XI

CONCLUSION AND SCOPE FOR FUTURE WORK

11.1 Conclusions:

In the present study various deterministic as well as stochastic models have been developed for two-item/ multi-item inventory system to take account of interaction of demand due to shortages and/or introduction of new item. The models take account of many realistic situations in inventory management. The solution procedure have been discussed for each model. Some numerical examples have been taken to examine the possibility of substitution.

For deterministic demand case, the model with independent procurement of two items has been developed. And its three cases have been dealt separately. A large number of problems have been solved to study parametric analysis. And analysis shows that in most of the cases, the total operating cost of inventory system is reduced when shortages of one item are substituted by the other. The model has been extended for the case of shortages allowed with partial backlogging. The model has been developed to study the effect of complete and partial substitution on the total annual operating cost, of jointly procured two item inventory system.

It has been further extended for the case of coordinated replenishment policy. The model for the case of continuous substitution has been developed to study the effect on total annual operating cost of jointly-procured-two-item inventory system by introduction of new product to substitute the demand of old product. Model for dynamic demand case has been developed to study the effect of holding, backlogging and substitution of the demand of one item by the other on total cost. The concept of dynamic programming has been used to solve the problem. The result obtained from numerical example shows that the substitutions of the demand of one item by other for some periods is preferable and sometimes, their independent procurement are preferred. The decision for the substitution depends on relative difference of various costs of two items. Various rules have been developed for complete substitution of the demand of one product by the other in multi-item inventory system.

For the case of stochastic demand, joint replenishment policy with substitution has been formulated for the popular periodic review (R, T) policy. The demands in the lead time are assumed to be normally distributed. For (Q, r, T) policy with coordinated replenishment of two item, simulation technique has been used to obtain decision variables. The operating inventory levels for two-item inventory system have been found substantially different from the case when

items have been treated independently, and substantial savings are possible.

In all above cases as might be expected the enormous computational efforts are required for the complete solution.

1.2 Scope of Further Work:

The present work on substitution of demand of one item by the other length approaches towards the realistic situation, is still far from complete in that much more is to be done to evolve an operating policy for the practical situation. In order to account for realistic situation, models developed in this thesis need to be extended to account for following situation.

1. Quantities discount are given.
2. Probabilistic lead time for the arrival of the products.
3. Multi-item inventory system where more than two items are available to substitute the demand of stock-out item. Some preference relation can be developed.
4. Two or more items are in out of stock situations at a time.
5. The nature of substitution factor is probabilistic.

In order to reduce computational complexities involved in the solution methodology of all the cases, some heuristic procedures need to be developed to make models as far as possible computationally simpler and the work can be extended in this direction.

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APPENDIX A

$$E_1 = \int_{k_1}^{\infty} \sigma_1(z_1-k_1)\Phi\left(-\frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2\right)\ (z_1)dz_1$$

and,

$$E_2 = \int_{k_1}^{\infty} \int_{-\frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2}^{k_2} \sigma_2(k_2-z_2)\ (z_2)\, dz_2\ (z_1)\, dz_1 \qquad (A.1)$$

Differentiating Eq. (A.1) with respect to k_1 by applying Leibznith rule we get,

$$\frac{\partial E_1}{\partial k_1} = \int_{k_1}^{\infty} \Big[-\sigma_1\Phi\left(-\frac{\sigma_1}{\sigma_2}(z_1-k_1)\right) + \sigma_1(z_1-k_1)\frac{\partial}{\partial k_1}\int_{-\infty}^{-\frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2} \phi(z_2)\, dz_2\Big]\ \phi(z_1)\, dz_1$$

$$0 - \sigma_1(k_1-k_1)\cdot\Phi\left(-\frac{\sigma_1}{\sigma_2}(k_1-k_1)+k_2\right)\phi(z_1)$$

$$= \int_{k_1}^{\infty} \Big[-\sigma_1\Phi\left[-\frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2\right] + \frac{\sigma_1^2}{\sigma_2}(z_1-k_1)$$

$$\phi\left[-\frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2\right]\phi(z_1)\, dz_1 \qquad (A.2)$$

$$\frac{\partial E_1}{\partial k_2} = \int_{k_1}^{\infty} \left[0 + 0 - \sigma_1(z_1-k_1)\ \phi\left(-\frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2\right)\phi(z_1)\right]dz_1$$

$$= -\sigma_1\int_{k_1}^{\infty}(z_1-k_1)\ \phi\left(-\frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2\right)\phi(z_1)dz_1 \qquad (A.3)$$

$$\frac{\partial E_2}{\partial k_1} = \int_{k_1}^{\infty} - \frac{\sigma_1}{\sigma_2} \cdot \sigma_2 (k_2 + \frac{\sigma_1}{\sigma_2}(z_1-k_1)-k_2)\, \phi\, (- \frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2)$$

$$\phi(z_1)\, dz_1 + 0 - \int_{k_2}^{k_2} + \phi(z_1)_2 (k_2 - z_2)\, \phi\,(z_2)\, dz_2$$

$$= - \frac{\sigma_1^2}{\sigma_2} \int_{k_1}^{\infty} (z_1-k_1)\; \phi(- \frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2)\phi(z_1) dz_1 \tag{A.4}$$

$$\frac{\partial E_2}{\partial k_2} = \int_{k_1}^{\infty} [\int_{-\frac{\sigma_1}{\sigma_2}[z_1-k_1]+k_2}^{k_2} \sigma_2\, \phi(z_2\, dz_2) + \sigma_2 (k_2-k_2)\, \phi\,(k_2)$$

$$+ \frac{\sigma_1}{\sigma_2}\; (z_1-k_1)\; \phi(- \frac{\sigma_1}{\sigma_2}(z_1-k_1)+k_2)]\, \phi\,(z_1) dz_1$$

$$= \int_{k_1}^{\infty} \sigma_2\, [\Phi\,(k_2) - \Phi(- \frac{\sigma_1}{\sigma_2}\, (z_1-k_1)+k_2)]\; \phi(z_1)\; dz_1$$

$$+ \sigma_1 \int_{k_1}^{\infty} (z_1-k_1) \quad (- \frac{\sigma_1}{\sigma_2}\, (z_1-k_1) + k_2)\; \phi(z_1)\; dz_1 \tag{A.5}$$

Since,

$$ES_{12} = E_1 + E_2 \tag{A.6}$$

and,

$$B^{(1)}(k_1,k_2) = \int_{k_1}^{\infty} \sigma_1 (z_1-k_1) \phi(z_1) dz_1 - E_1 - E_2 \tag{A.7}$$

From Eqs. (A.2) - (A.5), we get,

$$\frac{\partial ES_{12}}{\partial\, k_1} = \frac{\partial E_1}{\partial k_1} + \frac{\partial E_2}{\partial k_2} = \int_{k_1}^{\infty} \Phi\, (- \frac{\sigma_1}{\sigma_2}\, (z_1-k_1)+k_2)\; \phi(z_1)\; dz_1 \tag{A.8}$$

$$\frac{\partial ES_{12}}{\partial k_2} = \frac{\partial E_1}{\partial k_2} + \frac{\partial E_2}{\partial k_2} = \sigma_2 \Phi(k_2)\ [1 - \Phi(k_1)] - \sigma_2 \int_{k_1}^{\infty} \Phi[- \frac{\sigma_1}{\sigma_2}(z_1 - k_1) + k_2]\ \phi(z_1)\ dz_1 \qquad \text{(A.9)}$$

$$\frac{\partial B^{(1)}(k_1,k_2)}{\partial k_1} = - \int_{k_1}^{\infty} \sigma_1 \phi(z_1)\ dz_1 + 1 \int_{-k_1}^{\infty} \Phi[- \frac{\sigma_1}{\sigma_2}(z_1 - k_1) + k_2]\ \phi(z_1)\ dz_1$$

$$= \sigma_1\ [1 - \Phi(k_1)] + \sigma_1 \int_{k_1}^{\infty} \Phi\ [- \frac{\sigma_1}{\sigma_2}(z_1 - k_1) + k_2)\ \phi(z_1)\ dz \qquad \text{(A.10)}$$

$$\frac{\partial B^{(1)}(k_1,k_2)}{\partial k_2} = - \sigma_2 \Phi(k_2)\ [1 - \Phi(k_1)] + \sigma_2 \int_{k_1}^{\infty} \Phi\ [- \frac{\sigma_1}{\sigma_2}(z_1 - k_1) + k_2]\ \phi(z_1)\ dz_1 \qquad \text{(A.11)}$$